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Vol. XXXIII No. 10 October 2015

Corporate Office
Plot 99, Sector 44 Institutional Area, Gurgaon, (HR).
Tel : 0124-4951200
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Regd. Office
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110 029.

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# editorial

## The Secret of Success in Life is To be a Good Teacher

The relevance of this editorial is for every one who wants to be a good scientist, a great economist or a big person in politics.

The first quality of a great scientist is the ability to explain what he has done to other scientists and students. He should have the command of language to express without ambiguity so that he speaks what he wants to convey in a clear way, with the flair to help the audience spellbound. Simply speaking, he must be able to convey what is in his mind.

By writing long equations on the board, the attention of the audience will waver. Even a good meal requires some topping, at least a sprinkle of pepper and salt. The ability to make students or scientists roar with laughter is the only way of keeping the audience glued to their seats with all the attention only to the speaker. There is no substitute for hard work and knowing the subject thoroughly from historical notes to be latest theories.

Precision and perfection in thought, ability to verify experimental and theoretical work and being up to date in the subject whether it is law or medicine, science or vedanta is absolutely necessary for any teacher.

From carpenters to nuclear scientists, one will have to explain to others what has been done by them and teach what else has to be done. Take a few models, either pertaining to your subject or in totally different fields. A good teacher in any subject has a lot to teach us, directly or through their writings or research publications. It is up to us to wake up and win.

Anil Ahlawat
Editor

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# MATHS MUSING

*Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.*
*During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.*
*Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.*

**Prof. Dr. Ramanaiah Gundala,** Former Dean of Science and Humanities, Anna University, Chennai

## PROBLEM Set 154

### JEE MAIN

**1.** A fair coin is tossed 10 times. The probability of getting atleast 4 consecutive heads is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{2}{5}$

**2.** The difference of maximum and minimum values of the function $f(\theta) = \sin^2(\cos\theta) + \cos^2(\sin\theta)$ is

(a) $1 - \sin 1$ (b) $1 - \sin 2$

(c) $1 - \cos 1$ (d) $1 - \cos 2$

**3.** Consider all 5-digit numbers such that the sum of their digits is 7 and each of the digits 0, 1, 2 occurs atleast once in them, the number of such numbers is

(a) 96 (b) 100 (c) 102 (d) 104

**4.** Let $a < b < c$ be integers and $c$ is a multiple of $b$. If $a$, $b$, $c$ are in H.P. and $a = 20$, then the number of values of $b$ is

(a) 3 (b) 4 (c) 5 (d) 6

**5.** If $(x+y)\frac{dy}{dx} = 2$, then $x + y + 2 =$

(a) $ce^y$ (b) $ce^{y/2}$

(c) $ce^{-y}$ (d) $ce^{-y/2}$

### JEE ADVANCED

**6.** Let $t_1, t_2, t_3$ be the lengths of tangents from a point $P$ to the circles $x^2 + y^2 = a^2$, $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2ay$ respectively. If $t_1^4 = t_2^2 t_3^2 + 4$, then the locus of $P$ consists of

(a) a straight line (b) a circle

(c) a parabola (d) an ellipse

### COMPREHENSION

Let $f(x)$ be a differentiable function such that $f(x + y) = e^x f(y) + e^y f(x)$ for all $x$, $y$ and $f'(0) = 1$.

**7.** $f(x)$ has

(a) maximum (b) minimum

(c) both maximum and minimum

(d) neither maximum nor minimum

**8.** The area bounded by the curve $y = f(x)$ and the $x$-axis is

(a) 1 (b) $\frac{1}{2}$

(c) 2 (d) $e$

### INTEGER MATCH

**9.** In a triangle $ABC$, if $a = 3\sqrt{3}$ and $c = 2b$, then its maximum area is

### MATCHING LIST

**10.** The complex numbers $z_1, z_2, z_3, \ldots z_n$ represent the vertices of a regular polygon of $n$ sides and $z_0$ is its centre. Let $z_0 = Az_1 + \bar{A}z_2$ and $[x]$ be the greatest integer $\le x$.

| | $n$ | | $[\lvert A\rvert^2]$ |
|---|---|---|---|
| P. | 6 | 1. | 4 |
| Q. | 8 | 2. | 3 |
| R. | 10 | 3. | 2 |
| S. | 12 | 4. | 1 |

| | P | Q | R | S |
|---|---|---|---|---|
| (a) | 4 | 1 | 2 | 3 |
| (b) | 4 | 4 | 3 | 2 |
| (c) | 1 | 1 | 2 | 3 |
| (d) | 1 | 3 | 4 | 2 |

See Solution set of Maths Musing 153 on page no. 78

**Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series**

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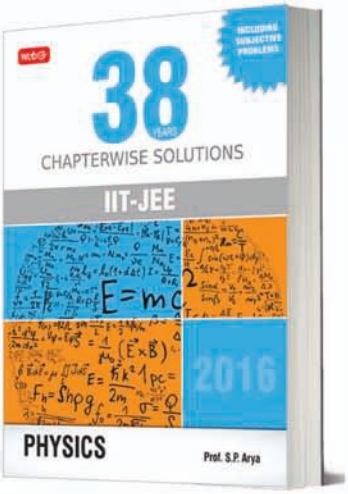


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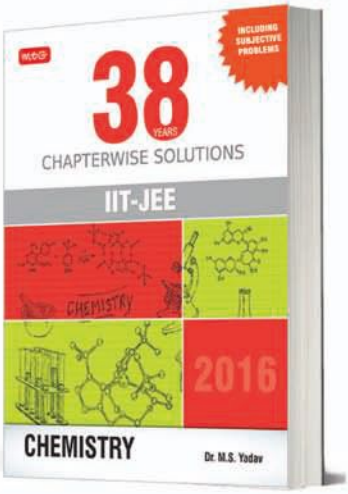


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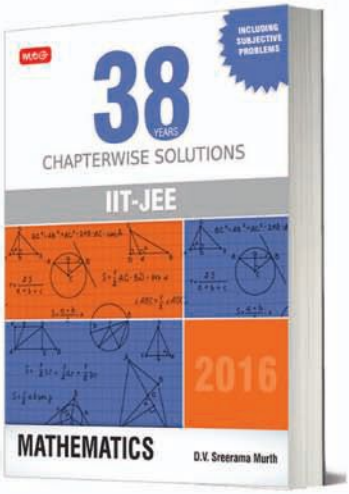


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# JEE WORK OUTS

PAPER-1

## SECTION-1

**This section contains EIGHT questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.**

**1.** Find the integer $n$ for which

$\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number.

**2.** Find the number of values of $x$ where

$f(x)=\begin{cases}x, & \text{if } x \text{ is rational}\\ 1-x, & \text{if } x \text{ is irrational}\end{cases}$ is continuous.

**3.** A function $f(x)$ is defined as

$f(x)=\begin{cases}x^m\sin\frac{1}{x}, & x\neq 0,\ m\in N\\ 0, & x=0\end{cases}$

The least value of $m$ for which $f'(x)$ is continuous at $x = 0$ is

**4.** If $f(x+y)=f(x)\cdot f(y)$, $f(0)\neq 0$ for all $x$, $y$ and $f(2)=2, f'(0)=3$, then $f'(2)$ equals

**5.** $f(x)=\frac{1}{1+|x|}+\frac{1}{1+|x-1|}$. Let $x_1$, $x_2$ be the points where $f(x)$ attains local minimum and global maximum respectively. Let $k=f(x_1)+f(x_2)$, then $6k-9=$ .

**6.** $f(x)=\begin{cases}\left(\sqrt{2}+\sin\frac{1}{x}\right)e^{\frac{-1}{|x|}} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$

Number of points where $f(x)$ has local extrema when $x \neq 0$ be $n_1$ and $n_2$ be the value of global minimum of $f(x)$, then $n_1 + n_2 =$

**7.** A flower bed is to be in the shape of a circular sector of radius $r$ and central angle $\theta$. If the area is fixed and perimeter is minimum, find the angle $\theta$ in radians.

**8.** Let $f(x) = [3 + 4\sin x]$ (where $[\cdot]$ denotes the greatest integer function). If sum of all the values of $x$ in $[\pi, 2\pi]$ where $f(x)$ fails to be differentiable, is $\frac{k\pi}{2}$, then the value of $k/8$ is

## SECTION-2

**This section contains TEN questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option (s) is (are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.**

**9.** A twice differentiable function $f(x)$ is defined for all real numbers and satisfies the following conditions: $f(0) = 2$; $f'(0) = -5$ and $f''(0) = 3$. The function $g(x)$ is defined by $g(x) = e^{ax} + f(x)$, $\forall\, x \in R$, where $a$ is any constant. If $g'(0) + g''(0) = 0$, then $a$ can be equal to

(a) 1 (b) –1 (c) 2 (d) –2

**10.** Let $f : R \to I$, $f(x) = [x]^2 + [x + 1] - 3$, where $[x]$ = the greatest integer $\le x$. Then

(a) $f(x)$ is a many-one and not onto function

(b) $f(x) = 0$ for infinite number of values of $x$

(c) $f(x) = 0$ for only two real values of $x$

(d) $f(x)$ is a many-one and onto function

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**11.** If $f(x)=\left(\frac{|x|}{|x|+2}\right)^{-x}$, then

(a) $\lim\limits_{x\to-\infty} f(x)=e^{-2}$ (b) $\lim\limits_{x\to-\infty} f(x)=0$

(c) $\lim\limits_{x\to1} f(x)=\frac{1}{3}$ (d) $\lim\limits_{x\to\infty} f(x)=e^{2}$

**12.** Let $f : R \to R$ be a function defined as $f(x)=x+[x]$. (Where $[x]$ denotes the greatest integer less than or equal to $x$). Which of the following hold(s) good?

(a) $f(x)$ is a periodic

(b) $f(x)$ is surjective

(c) $f(x)$ is neither odd nor even

(d) $f(x)$ is injective

**13.** If $[x]$ and $\{x\}$ denotes the greatest integer function less than or equal to $x$ and fractional part function, then the number of real $x$, satisfying the equation $(x-2)[x]=\{x\}-1$, is

(a) 0 (b) 1

(c) 2 (d) infinite

**14.** Domain of $f(x)=\sin^{-1}[2-4x^2]$ is ($[\cdot]$ denotes the greatest integer function)

(a) $\left[-\frac{\sqrt3}{2},\frac{\sqrt3}{2}\right]-\{0\}$ (b) $\left[-\frac{\sqrt3}{2},\frac{\sqrt3}{2}\right]$

(c) $\left[-\frac{\sqrt3}{2},0\right)$ (d) $\left[-\frac{\sqrt3}{2},8\right]$

**15.** For every pair of continuous functions $f, g : [0, 1] \to R$ such that

$\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0, 1]\}$,

the correct statement(s) is (are)

(a) $(f(c))^2+3f(c)=(g(c))^2+3g(c)$ for some $c \in [0, 1]$

(b) $(f(c))^2+f(c)=(g(c))^2+3g(c)$ for some $c \in [0, 1]$

(c) $(f(c))^2+3f(c)=(g(c))^2+g(c)$ for some $c \in [0, 1]$

(d) $(f(c))^2=(g(c))^2$ for some $c \in [0, 1]$

**16.** If the normal to $y=f(x)$ at $(0, 0)$ is given by $y-x=0$, then

$\lim\limits_{x\to0}\frac{x^2}{f(x^2)-20f(9x^2)+2f(99x^2)}$ is equal to

(a) $\frac{1}{19}$ (b) $-\frac{1}{19}$

(c) $\frac{1}{2}$ (d) does not exist

**17.** In which of the following cases the given equations has atleast one root in the indicated interval?

(a) $x-\cos x=0$ in $(0, \pi/2)$

(b) $x+\sin x=1$ in $(0, \pi/6)$

(c) $\frac{a}{x-1}+\frac{b}{x-3}=0,\ a,b>0$ in $(1, 3)$

(d) $f(x)-g(x)=0$ in $(a, b)$ where $f$ and $g$ are continuous on $[a, b]$ and $f(a) > g(a)$ and $f(b) < g(b)$.

**18.** Let $f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x)=\frac{\pi}{2}\sin x$ for all $x \in R$. Then which of the following is (are) true?

(a) Range of $f$ is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(b) Range of $fog$ is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(c) $\lim\limits_{x\to0}\frac{f(x)}{g(x)}=\frac{\pi}{6}$

(d) There is an $x \in R$ such that $(gof)(x)=1$

## SECTION-3

**This section contains TWO questions. Each question contains two columns, Column I and Column II. Column I has four entries (A), (B), (C) and (D), Column II has five entries (P), (Q), (R) and (S). Match the entries in Column I with the entries in Column II. One or more entries in Column I may match with one or more entries in Column II. The ORS contains a 4 × 5 matrix whose layout will be similar to the one shown below.**

(A) P Q R S

(B) P Q R S

(C) P Q R S

(D) P Q R S

**For each entry in Column I, darken the bubbles of all the matching entries. For example, if entry (A) in Column I matches with entries (Q) and (R), then darken these two bubbles in the ORS. Similarly, for entries (B), (C) and (D). Marking scheme : For each entry in Column I +2 If only the bubble(s) corresponding to all the correct match(es) is (are) darkened, 0 If none of the bubbles is darkened, –1 In all other cases.**

**19.** Let the functions defined in column I have domain $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and range $(-\infty, \infty)$.

| Column I | | Column II | |
|---|---|---|---|
| (A) | $1+2x$ | P. | onto but not one-one |
| (B) | $\tan x$ | Q. | one-one but not onto |
| (C) | $x^2$ | R. | one-one and onto |
| (D) | $\tan(x^2)$ | S. | neither one-one nor onto |

**20.** Match the following:

| Column I | | Column II | |
|---|---|---|---|
| (A) | Number of integral values of $x$ which lie in the range of the function $f(x)=\sqrt{x^2+4x}C_{2x^2+3}$ is | P. | 1 |
| (B) | Number of integral values of $x$ which lie in the range of the function $f(x)=\frac{\sin x}{\sqrt{1+\tan^2 x}}-\frac{\cos x}{\sqrt{1+\cot^2 x}}$ is | Q. | 2 |
| (C) | Period of the function $f(x)=\sin(\sin(\pi x))+e^{x-[x]}$ where $[x]$ denotes the greatest integer less than or equal to $x$ is | R. | 3 |
| (D) | Number of ways of dividing the number 360, in two factors which are prime to each other, is equal to | S. | 4 |

## PAPER-2

### SECTION-1

**This section contains EIGHT questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.**

**1.** Number of functions $f:\{a, b, c\}\to\{1, 2, 3, 4, 5, 6\}$ which are not injective is equal to $12n$. Find the value of $n$.

**2.** Let $f(x, y)$ be a periodic function satisfying the condition $f(x, y)=f((2x+2y),(2y-2x))\ \forall\ x, y\in R$. Now, define a function $g$ by $g(x)=f(2^x, 0)$. If period of $g(x)$ function is $\alpha$, then value of $|\alpha-7|$ is

**3.** If $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)$ for all $x\in R$, then the value $f(1)=$

**4.** If the curves $ay+x^2=7$ and $y=x^3$ cut each other orthogonally at a point, then the value of $|a|$ is

**5.** For non-negative integers $m$, $n$ define a function as follows:

$$f(m,n)=\begin{cases}n+1 & \text{if}\quad m=0\\ f(m-1,1) & \text{if}\quad m\neq 0,\ n=0\\ f\big((m-1),f(m,n-1)\big) & \text{if}\quad m\neq 0,\ n\neq 0\end{cases}$$

Then the value of $f(1, 1)$ is

**6.** $f(x)=\lim\limits_{t\to\infty}\dfrac{(a+\sin\pi x)^t-1}{(a+\sin\pi x)^t+1}$, $x\in(0, 6)$, then the number of points of discontinuity of $f(x)$ when $a=1$ is

**7.** If the function $f(x)=x^3+e^{x/2}$ and $g(x)=f^{-1}(x)$, then the value of $g'(1)$ is

**8.** Let $f: R\to R$, defined as,

$$f(x)=\frac{x^4}{x^8+2x^6-4x^4+8x^2+16}.$$

Also, let $\alpha$ and $\beta$ be the maximum and minimum values, respectively, of $f(x)$. Then $12(\alpha-\beta)=$ ____

### SECTION-2

**This section contains EIGHT questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option (s) is (are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.**

**9.** Let $f: R\to(1, 3)$ is defined by $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}+2$ then

(a) $f^{-1}(x)$ does not exist due to many-one

(b) $f^{-1}(x)$ does not exist due to not onto function

(c) invertible function

(d) $f^{-1}(x)=\dfrac{1}{2}\log\left(\dfrac{x-1}{3-x}\right)$

**10.** Given $f(x) = -\frac{x^3}{3} + x^2 \sin 1.5a - x \sin a \cdot \sin 2a - 5\sin^{-1}(a^2 - 8a + 17)$, then

(a) $f'(x) = -x^2 + 2x\sin 6 - \sin 4 \sin 8$

(b) $f'(\sin 8) > 0$

(c) $f'(x)$ is not defined at $x = \sin 8$

(d) $f'(\sin 8) < 0$

**11.** Nick has 3 children by his first wife. Amy has $x$ children by her first husband. They marry and have children of their own. The whole family has 12 children. Assume that two children of same parents do not fight and the fight is always between exactly two children. The value of $x$ for which the maximum number of fights occur is / are

(a) 6 (b) 4 (c) 8 (d) 5

**12.** If $x^y = e^{x-y}$, then $\frac{dy}{dx} =$

(a) $\log x(1 + \log x)^{-2}$ (b) $(1 + \log x)^{-2}$

(c) $(1 + \log x)^{-1}$ (d) $\log x(1 + \log x)^{-1}$

**13.** Let $f(x) = \begin{cases} x^\alpha \log x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

If Rolle's theorem can be applied to $f$ on $[0, 1]$, then value of $\alpha$ can be

(a) $-1$ (b) $-1/2$ (c) 0 (d) $1/2$

**14.** Let $f(x) = [x]$ where $[t]$ is the greatest integer not exceeding $t$ and $g(x) = |x|$. Then $(fog)(x)$ is

(a) differentiable at $x = -1$ and $(fog)'(-1) = 1$

(b) differentiable at $x = 0$ and $(fog)'(0) = -1$

(c) differentiable at $x = 0$ and $(fog)'(0) = 0$

(d) not differentiable at $x = -1$

**15.** Which of the following are true?

(a) The function $f(x) = x^5 + 2x^3 + x$ increases in $(-\infty, \infty)$.

(b) The function $g(x) = x^3 + 2x - 5$ increase

(c) The function $h(x) = 2x/\log_e x$ decreases in $(0, 1) \cup (1, e)$ and increases in $(e, +\infty)$.

(d) The function $\phi(x) = x^3 - ax$ increases on the entire line if $a \le 0$.

**16.** The function $y = f(x)$ is parametrically represented by the equations

$x = t^5 + 5t^3 - 30t + 7$

$y = 4t^3 - 3t^2 - 18t + 3$

where $|t| < 2$. Then $y = f(x)$ has maximum value 14 at $x$ equals

(a) 21 (b) 11 (c) 31 (d) $-21$

## SECTION-3

**This section contains TWO paragraphs. Based on each paragraph, there will be TWO questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme :+4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.**

### Paragraph for question No. 17 and 18

Let $f(x) = \log_{\{x\}}[x]$, $g(x) = \log_{[x]}\{x\}$,
(where $[\cdot]$, $\{\cdot\}$ denotes greatest integer function and fractional part).

**17.** For $x \in (1, 5)$, $f(x)$ is not defined at how many points?

(a) 5 (b) 4 (c) 3 (d) 2

**18.** If $A = \{x: x \in$ domain of $f(x)\}$ and $B = \{x : x \in$ domain of $g(x)\}$, then $A - B$ will be

(a) (2, 3) (b) (1, 3)

(c) (1, 2) (d) none of these

### Paragraph for Q. No. 19 and 20

An even periodic function $f : R \to R$ with period 4 is such that

$$f(x) = \begin{cases} \max(|x|, x^2); & 0 \le x < 1 \\ x \quad ; & 1 \le x \le 2 \end{cases}$$

**19.** The value of $\{f(x)\}$ at $x = 5.12$ (where $\{\ \}$ represents fractional part), is

(a) $\{f(7.88)\}$ (b) $\{f(3.26)\}$

(c) $\{f(2.12)\}$ (d) $\{f(5.88)\}$

**20.** If $g(x) = |3 \sin x|$, then the number of solutions of $f(x) = g(x)$ for $x \in (-6, 6)$, are

(a) 5 (b) 7 (c) 3 (d) 9

## ANSWER KEYS

### PAPER-1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | (3) | 2. | (1) | 3. | (3) | 4. | (6) | 5. | (8) |
| 6. | (0) | 7. | (2) | 8. | (3) | 9. | (a, d) | 10. | (a, b) |
| 11. | (a, d) | 12. | (c, d) | 13. | (d) | 14. | (a) | 15. | (a, d) |
| 16 | (b) | 17. | (a, b, c, d) | | | 18. | (a, b, c) | | |

19. (A) → (Q); (B) → (R); (C) → (S); (D) → (S)

20. (A) → (P), (B) → (R), (C) → (Q), (D) → (S)

### PAPER-2

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | (8) | 2. | (5) | 3. | (4) | 4. | (6) | 5. | (3) |
| 6. | (5) | 7. | (2) | 8. | (1) | 9. | (c, d) | 10. | (a, d) |
| 11. | (b, d) | 12. | (a) | 13. | (d) | 14. | (c, d) | | |
| 15. | (a, b, c, d) | | | 16. | (c) | 17. | (c) | 18. | (c) |
| 19. | (a) | 20. | (b) | | | | | | |

***For detailed solution, visit our website www.vidyalankar.org***

# BIHAR CECE

SECOND STAGE

## SOLVED PAPER 2015

**1.** The range of function $f(x) = {}^{7-x}P_{x-3}$ is

(a) {1, 2, 3, 4} (b) {0, 1, 2, 3, 4, 5}

(c) {0, 1, 2, 3} (d) {1, 2, 3, 4, 5, 6}

**2.** The number of ways in which we can post 5 letters in 10 letter boxes is

(a) 50 (b) $5^{10}$ (c) $10^5$ (d) ${}^{10}C_5$

**3.** The product of first $n$ odd natural numbers is equal to

(a) $({}^{2n}C_n)({}^nP_n)$ (b) $\frac{1}{2}({}^{2n}C_n)({}^nP_n)$

(c) $\frac{1}{2^n}({}^{2n}C_n)({}^nP_n)$ (d) $\frac{1}{2^{2n}}({}^{2n}C_n)({}^nP_n)$

**4.** If $n \in N$, then the remainder when $37^{n+2} + 16^{n+1} + 30^n$ is divided by 7 is

(a) 0 (b) 1 (c) 2 (d) 5

**5.** If the fifth term in the expansion of $\left(x^{1/3} + \frac{1}{x}\right)^n$ does not depend on $x$, then $n$ is equal to

(a) 8 (b) 12

(c) 16 (d) 20

**6.** Given $A = \begin{bmatrix} 0 & -\tan\alpha \\ \tan\alpha & 0 \end{bmatrix}$

and $B(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then

$(I + A)(I - A)^{-1}$ is equal to

(a) $B(\alpha)$ (b) $B(-\alpha)$ (c) $B(2\alpha)$ (d) $B(-2\alpha)$

**7.** If the system of equations $x + 2y - 3z = 1$, $(p + 2)z = 3$, $(2p + 1)y + z = 2$ has infinite number of solutions, then the value of $p$ is not equal to

(a) $-2$ (b) $-\frac{1}{2}$ (c) 0 (d) 2

**8.** If $\alpha$, $\beta$ and $\gamma$ are real numbers, then

$\Delta = \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$ is equal to

(a) 1 (b) $-1$

(c) $\cos\alpha\cos\beta\cos\gamma$ (d) 0

**9.** Two dice are thrown simultaneously. What is the probability of getting two numbers whose product is even?

(a) $\frac{1}{4}$ (b) $\frac{3}{4}$ (c) $\frac{5}{12}$ (d) $\frac{7}{12}$

**10.** A box contains 20 electric bulbs, out of which 4 are defective. Two bulbs are chosen at random from this box. The probability that at least one of these is defective is

(a) $\frac{4}{19}$ (b) $\frac{5}{19}$ (c) $\frac{6}{19}$ (d) $\frac{7}{19}$

**11.** If $x, y \in R$ and $x^3 + y^3 = 16$, then the maximum possible value of $(x + y)$ is

(a) 2 (b) 4 (c) 8 (d) 12

**12.** If $a, b > 0$, $a + b = 1$, then least value of $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)$ is

(a) 3 (b) 6 (c) 9 (d) 12

**13.** If $m$ is a natural number such that $m \le 5$, then the probability that the quadratic equation $x^2 + mx + \frac{1}{2} + \frac{m}{2} = 0$ has real roots is

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{1}{3}$

**14.** If $\sin x + \sin^2 x = 1$, then the value of $(\cos^{12}x + 3\cos^{10}x + 3\cos^8 x + \cos^6 x - 1)$ is equal to

(a) 2 (b) 1 (c) $-1$ (d) 0

***Contributed by*** : Paradise Institute, Patna

**15.** In a triangle $ABC$, right angled at $C$, $(\tan A + \tan B)$ is equal to

(a) $a + b$ (b) $\frac{c^2}{ab}$

(c) $\frac{a^2}{bc}$ (d) $\frac{b^2}{ac}$

**16.** If the area of a triangle $ABC$ is given by $\Delta = a^2 - (b - c)^2$, then $\tan\frac{A}{2}$ is equal to

(a) $-1$ (b) $0$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

**17.** In a triangle $ABC$, if $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$, then angle $C$ is equal to

(a) $30°$ (b) $60°$ (c) $90°$ (d) $120°$

**18.** $\begin{vmatrix}\cos(\theta+\alpha) & \sin(\theta+\alpha) & 1\\ \cos(\theta+\beta) & \sin(\theta+\beta) & 1\\ \cos(\theta+\gamma) & \sin(\theta+\gamma) & 1\end{vmatrix}$ is independent of

(a) $\alpha$ (b) $\beta$ (c) $\gamma$ (d) $\theta$

**19.** $(m + 2)\sin\theta + (2m - 1)\cos\theta = (2m + 1)$, if $\tan\theta$ is equal to

(a) $\frac{4}{3}$ or $\frac{2m}{m^2-1}$ (b) $\frac{3}{4}$ or $\frac{2m}{m^2+1}$

(c) $\frac{4}{3}$ or $\frac{2m+1}{m^2}$ (d) $\frac{3}{4}$ or $\frac{m^2}{2m+1}$

**20.** If $0 < \alpha, \beta < \pi$ and $\cos\alpha + \cos\beta - \cos(\alpha + \beta) = \frac{3}{2}$, then

(a) $\alpha=\beta=\frac{2\pi}{3}$ (b) $\alpha=\frac{\pi}{3},\beta=\frac{2\pi}{3}$

(c) $\alpha=\frac{2\pi}{3},\beta=\frac{\pi}{3}$ (d) $\alpha=\beta=\frac{\pi}{3}$

**21.** A tower stands at the centre of a circular park. $A$ and $B$ are two points on the boundary of the park such that $AB$ $(= a)$ subtends an angle of $60°$ at the foot of the tower and the angle of elevation of the top of the tower from $A$ or $B$ is $30°$. The height of the tower is

(a) $2a\sqrt{3}$ (b) $a/\sqrt{3}$

(c) $a\sqrt{3}$ (d) $2a/\sqrt{3}$

**22.** If $4\sin^4 x + \cos^4 x = 1$, then $x$ is equal to

(a) $\frac{n\pi}{2}$ (b) $n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}}$

(c) $\frac{2n\pi}{3}$ (d) $2n\pi \pm \frac{\pi}{4}$

**23.** The solution of

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ is

(a) $n\pi+\frac{\pi}{8}$ (b) $\frac{n\pi}{2}+\frac{\pi}{8}$

(c) $(-1)^n\frac{n\pi}{2}+\frac{\pi}{8}$ (d) $2n\pi+\frac{\pi}{8}$

**24.** The value of $\cos^{-1}x+\cos^{-1}\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right)$, $\left(\frac{1}{2}\le x\le 1\right)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\pi$ (d) zero

**25.** The function $\sin[\cot^{-1}(x + 1)] - \cos(\tan^{-1}x) = 0$ for $x$ is equal to

(a) $-\frac{1}{2}$ (b) $0$ (c) $\frac{1}{2}$ (d) $1$

**26.** The maximum value of $(2997\sin x + 3996\cos x)$ is equal to

(a) 2997 (b) 3996 (c) 4995 (d) 6993

**27.** Reflection of the line $x + y + 1 = 0$ in the line $2x + y + 1 = 0$ is

(a) $x + 7y - 11 = 0$ (b) $7x + y + 1 = 0$

(c) $7x + y - 11 = 0$ (d) $7x + y + 7 = 0$

**28.** The area of the triangle formed by the lines $y = ax$, $x + y = a$ and the $y$-axis is equal to

(a) $\frac{1}{2|1+a|}$ (b) $\frac{a^2}{|1+a|}$

(c) $\frac{1}{2}\frac{a}{|1+a|}$ (d) $\frac{a^2}{2|1+a|}$

**29.** A stick of length $l$ rests against the floor and a wall of a room. If the stick begins to slide on the floor, then the locus of its middle point is

(a) a straight line (b) a circle

(c) a parabola (d) an ellipse

**30.** $P$ is a point on the line $y + 2x = 1$ and $Q$ and $R$ are two points on the line $3y + 6x = 6$ such that triangle $PQR$ in an equilateral triangle. The length of the side of the triangle is

(a) $\frac{1}{\sqrt{15}}$ (b) $\frac{2}{\sqrt{15}}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{4}{\sqrt{5}}$

**31.** Two tangents to the circle $x^2 + y^2 = 4$ at the points $A$ and $B$ meet at point $P(-4, 0)$. The area of the quadrilateral $PAOB$ in sq. units, when $O$ is origin, is

(a) $4\sqrt{2}$ (b) $6\sqrt{2}$ (c) $4\sqrt{3}$ (d) $8\sqrt{3}$

**32.** If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to another circle with centre (2, 1), then radius of this circle is

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 3 (d) 2

**33.** If (2, –8) is at one end of a focal chord of the parabola $y^2 = 32x$, then the other end of the chord is

(a) (2, –8) (b) (2, 8)
(c) (32, 32) (d) (32, –32)

**34.** Equation of common tangent of $y^2 = 4ax$ and $x^2 = 4ay$ is

(a) $x + y - a = 0$ (b) $x - y - a = 0$
(c) $x + y + a = 0$ (d) $x - y + a = 0$

**35.** The slope of a common tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and a concentric circle of radius $r$ is

(a) $\tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$ (b) $\sqrt{\frac{r^2-b^2}{a^2-r^2}}$

(c) $\left(\frac{r^2-b^2}{a^2-r^2}\right)$ (d) $\sqrt{\frac{a^2-r^2}{r^2-b^2}}$

**36.** The normal at an end of a latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through an end of the minor axis if

(a) $e^4 + e^2 = 1$ (b) $e^3 + e^2 = 1$
(c) $e^2 + e = 1$ (d) $e^3 + e = 1$

**37.** The equation $\left|\sqrt{x^2+(y-1)^2}-\sqrt{x^2+(y+1)^2}\right|=K$ will represent a hyper parabola for

(a) $K \in (0, 2)$ (b) $K \in (-2, 1)$
(c) $K \in (1, \infty)$ (d) $K \in (0, \infty)$

**38.** If the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, then the value of $b^2$ is

(a) 1 (b) 5 (c) 7 (d) 9

**39.** Given $A$ and $B$ as two sets, then $(A \cup B)^c \cup (A^c \cap B)$ is equal to

(a) $A^c$ (b) $B^c$ (c) $A$ (d) $B$

**40.** In a college of 300 students, every student reads five newspapers and every newspaper is read by 60 students. The total number of newspaper is

(a) at least 30 (b) at most 20
(c) exactly 25 (d) exactly 5

**41.** The range of $f(x)=\sec\left(\frac{\pi}{4}\cos^2 x\right)$, $-\infty < x < \infty$ is

(a) $[1, \sqrt{2}]$ (b) $[1, \infty)$
(c) $[-\sqrt{2}, 1] \cup [1, \sqrt{2}]$ (d) $(-\infty, -1] \cup [1, \infty)$

**42.** Given the function $f(x)=\frac{a^x+a^{-x}}{2}, a>2$, then $f(x + y) + f(x - y)$ is equal to

(a) $2f(x)\, f(y)$ (b) $f(x)\, f(y)$

(c) $\frac{f(x)}{f(y)}$ (d) $\frac{1}{2}f(x)\, f(y)$

**43.** The period of the function $|\sin x| + |\cos x|$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\pi$ (d) not defined

**44.** The inverse of the function $y=\frac{16^x-16^{-x}}{16^x+16^{-x}}$ is

(a) $\log_{16}(2 - x)$ (b) $\frac{1}{2}\log_{16}\frac{1+x}{1-x}$

(c) $\frac{1}{2}\log_{16}(2x-1)$ (d) $\frac{1}{4}\log_e\frac{2x}{2-x}$

**45.** $\lim\limits_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$ is equal to

(a) 2 (b) $\frac{1}{3}$ (c) –1 (d) $\frac{1}{2}$

**46.** If the function

$$f(x)=\begin{cases}x^2-(A+2)x+A & \text{for } x\neq 2\\ 2 & \text{for } x=2\end{cases}$$ is

continuous at $x = 2$, then '$A$' is equal to

(a) 2 (b) 1 (c) –1 (d) 0

**47.** If $\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\log a$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y}{x}$ (b) $\frac{x}{y}$ (c) $\frac{x^2}{y^2}$ (d) $\frac{y^2}{x^2}$

**48.** If $y=e^{\sqrt{x}}+e^{-\sqrt{x}}$, then $\left(x\frac{d^2y}{dx^2}+\frac{1}{2}\frac{dy}{dx}\right)$ is equal to

(a) $y$ (b) $xy$ (c) $\frac{1}{4}y$ (d) $\sqrt{x}y$

**49.** The points on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical are

(a) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$

(c) (0, 0) (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

**50.** A given right circular cone has a volume $p$. The largest right circular cylinder that can be inscribed in the cone has a volume $q$. Then $p : q$ is

(a) 8 : 3 (b) 9 : 4

(c) 7 : 2 (d) 3 : 2

**51.** If $I = \int \frac{dx}{\sqrt{(1-x)(x-2)}}$, then $I$ is equal to

(a) $\sin^{-1}(2x - 3) + C$ (b) $\sin^{-1}(2x + 5) + C$

(c) $\sin^{-1}(3 - 2x) + C$ (d) $\sin^{-1}(5 - 2x) + C$

**52.** If $I = \int \frac{\sin 2x}{(3+4\cos x)^3}\,dx$, then $I$ is equal to

(a) $\frac{3\cos x + 8}{(3+4\cos x)^2} + C$ (b) $\frac{3+8\cos x}{16(3+4\cos x)^2} + C$

(c) $\frac{3+\cos x}{(3+4\cos x)^2} + C$ (d) $\frac{3-8\cos x}{16(3+4\cos x)^2} + C$

**53.** If $I = \int_{1/e}^{e} |\log x| \frac{dx}{x^2}$, then $I$ is equal to

(a) 2 (b) $\frac{2}{e}$

(c) $2\left(1-\frac{1}{e}\right)$ (d) 0

**54.** If $I_1 = \int_0^{\pi/2} f(\sin 2x)\sin x\,dx$ and

$I_2 = \int_0^{\pi/4} f(\cos 2x)\cos x\,dx$, then $I_1/I_2$ is equal to

(a) 1 (b) 2 (c) $1/\sqrt{2}$ (d) $\sqrt{2}$

**55.** $\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1^2}} + \frac{1}{\sqrt{4n^2-2^2}} + \ldots\ldots + \frac{1}{\sqrt{4n^2-n^2}}\right]$ is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

**56.** The area enclosed between the curve $y = \log_e(x + e)$ and the coordinate axis is

(a) 4 (b) 3 (c) 2 (d) 1

**57.** The degree of the differential equation $y_2^{3/2} - y_1^{1/2} - 4 = 0$ is

(a) 2 (b) 3 (c) 4 (d) 6

**58.** The order of the differential equation $1 + \left(\frac{dy}{dx}\right)^5 = \frac{d^3y}{dx^3}$ is

(a) 5 (b) 3 (c) 1 (d) 0

**59.** The solution of $\frac{xdy}{x^2+y^2} = \left(\frac{y}{x^2+y^2} - 1\right)dx$ is

(a) $y = x\sin(c - x)$ (b) $y = x\cos(c - x)$

(c) $y = x\tan(c - x)$ (d) $y = x\cot(c - x)$

**60.** If $\left(\frac{2+\sin x}{y+1}\right)\frac{dy}{dx} = -\cos x$, $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to

(a) $-\frac{\pi}{3}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 1

**61.** Tangent to a curve intercepts the $y$-axis at a point $P$. A line perpendicular to this tangent through $P$ passes through another point (1, 0). The differential equation of the curve is

(a) $y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 1$ (b) $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

(c) $y\frac{dx}{dy} + x = 1$ (d) $x\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 1$

**62.** If $\vec{a}\cdot\hat{i} = \vec{a}\cdot(\hat{i}+\hat{j}) = \vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})$, then $\vec{a}$ is equal to

(a) $\hat{i}+\hat{j}+\hat{k}$ (b) $\hat{i}$

(c) $\hat{j}$ (d) $\hat{k}$

**63.** In a right angled triangle $ABC$, the hypotenuse $AB = p$. Then, $(\overrightarrow{AB}\cdot\overrightarrow{AC} + \overrightarrow{BC}\cdot\overrightarrow{BA} + \overrightarrow{CA}\cdot\overrightarrow{CB})$ is equal to

(a) $3p^2$ (b) $\frac{3p^2}{2}$ (c) $p^2$ (d) $\frac{p^2}{2}$

**64.** A unit vector coplanar with $(\hat{i}+\hat{j}+2\hat{k})$ and $(\hat{i}+2\hat{j}+\hat{k})$ and perpendicular to $(\hat{i}+\hat{j}+\hat{k})$ is

(a) $\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}$ (b) $\frac{(\hat{i}+\hat{j}+2\hat{k})}{\sqrt{6}}$

(c) $\frac{(\hat{i}+2\hat{j}+\hat{k})}{\sqrt{6}}$ (d) $\frac{(\hat{j}-\hat{k})}{\sqrt{2}}$

**65.** The magnitude of the torque exerted at origin by the force $\vec{F} = \hat{i} + 2\hat{j} + \hat{k}$ at the point (1, 1, 1) is

(a) 1 unit (b) $\sqrt{5}$ units

(c) $\sqrt{3}$ units (d) $\sqrt{2}$ units

**66.** The acute angle between the lines $x = -2 + 2t$, $y = 3 - 4t$, $z = -4 + t$ and $x = -2 - t$, $y = 3 + 2t$, $z = -4 + 3t$ is

(a) $\sin^{-1}\frac{1}{\sqrt{3}}$ (b) $\cos^{-1}\frac{1}{\sqrt{6}}$

(c) $\cos^{-1}\frac{1}{\sqrt{5}}$ (d) $\cos^{-1}\frac{2}{3}$

**67.** Length of the perpendicular from the point (−6, 2, 3) on the plane $3x - 6y + 2z + 10 = 0$ is

(a) 2 (b) $\frac{13}{7}$ (c) $\frac{10}{7}$ (d) $\frac{8}{7}$

**68.** A plane passes through (1, 2, 2) and is perpendicular to two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$. The square of the distance of the plane from the point (52, 53, 57) is

(a) 51 (b) 102 (c) 153 (d) 5202

**69.** If $d$ is the distance between the point (−1, −5, −10) and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ with the plane $x - y + z = 5$, then $d$ is equal to

(a) 5 (b) 12 (c) 13 (d) 17

**70.** Projection of point $P(\vec{p})$ on the plane $\vec{r}\cdot\vec{n} = q$ is $S(\vec{s})$, then $\vec{s}$ is equal to

(a) $\frac{(q - \vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$ (b) $\vec{p} + \frac{(q - \vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

(c) $\vec{p} - \frac{(\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$ (d) $\vec{p} - \frac{(q - \vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

**71.** If the algebraic sum of deviations of 20 observations from 30 is 20, then the mean of observations is

(a) 30 (b) 30.1 (c) 29 (d) 31

**72.** The median of a set of 9 distinct observations is 20.5. If each of the largest 4 observations of the set is increased by 2, then median of new set is

(a) increased by 2

(b) decreased by 2

(c) the same i.e. 20.5

(d) 2 times the original median

**73.** If the standard deviation of 0, 1, 2, 3, ....., 9 is $K$, then the standard deviation of 10, 11, 12, 13, ....., 19 is

(a) $K$ (b) $K + 10$

(c) $K + \sqrt{10}$ (d) $10K$

**74.** The variance of the first $n$ natural numbers is

(a) $\frac{(n^2 - 1)}{12}$ (b) $\frac{(n^2 - 1)}{6}$

(c) $\frac{(n^2 + 1)}{6}$ (d) $\frac{(n^2 + 1)}{12}$

**75.** The probability that a man will live 10 more years is $\frac{1}{4}$ and the probability that his wife will live 10 more years is $\frac{1}{3}$. Then the probability that neither will be alive in 10 years is

(a) $\frac{5}{12}$ (b) $\frac{1}{2}$ (c) $\frac{7}{12}$ (d) $\frac{2}{3}$

**76.** Forces acting on a particle have magnitudes of 5, 3 and 1 units and act in the direction of the vectors $6\hat{i} + 2\hat{j} + 3\hat{k}$, $3\hat{i} - 2\hat{j} + 6\hat{k}$ and $2\hat{i} - 3\hat{j} - 6\hat{k}$ respectively. They remain constant while the particle is displaced from the point $A(2, -1, -3)$ to $B(5, -1, 1)$. The work done is equal to

(a) 31 units (b) 33 units

(c) 34 units (d) 44 units

**77.** If $\vec{a}$ and $\vec{b}$ are two unit vectors, then the vector $(\vec{a} + \vec{b}) \times (\vec{a} \times \vec{b})$ is parallel to the vector

(a) $\vec{a} + \vec{b}$ (b) $2\vec{a} + \vec{b}$

(c) $\vec{a} - \vec{b}$ (d) $2\vec{a} - \vec{b}$

**78.** If the slope of one of the lines represented by $ax^2 - 6xy + y^2 = 0$ is twice the other, then '$a$' is equal to

(a) 1 (b) 2 (c) 4 (d) 8

**79.** If $y = mx$ bisects the angle between the lines $x^2(\tan^2\theta + \cos^2\theta) + 2xy\tan\theta - y^2\sin^2\theta = 0$ when $\theta = \frac{\pi}{3}$, then the value of $m$ is

(a) $\frac{-2 \pm \sqrt{7}}{3}$ (b) $\frac{-7 \pm \sqrt{2}}{3}$

(c) $\frac{-3 \pm \sqrt{7}}{2}$ (d) $\frac{-2 \pm \sqrt{3}}{4}$

**80.** $C_1 : x^2 + y^2 + 2x - 3 = 0$, $C_2 : x^2 + y^2 - 2x - 3 = 0$, $C_3 : x^2 + y^2 - 2y - 3 = 0$ are three circles. Which of the following statements is wrong?

(a) $C_1$ passes through the centre of $C_2$.

(b) $C_2$ passes through the centre of $C_1$.

(c) Centres of three circles form an equilateral triangle.
(d) Sum of the radii of $C_1$ and $C_2$ is twice the radius of $C_3$.

**81.** Let $f(x)=\begin{cases}1-\sqrt{1-x^2}, & \text{if } -1\le x\le 1\\ 1+\log\dfrac{1}{x}, & \text{if } x>1\end{cases}$

The function $f(x)$ is
(a) continuous and differentiable at $x = 1$
(b) not continuous but differentiable at $x = 1$
(c) continuous but not differentiable at $x = 1$
(d) neither continuous nor differentiable at $x = 1$

**82.** The least value of $k$ for which the function $x^2 + kx + 1$ is an increasing function in the interval $1 < x < 2$ is
(a) $-4$ (b) $-3$ (c) $-1$ (d) $-2$

**83.** The domain of definition of

$$f(x)=\log_2\left[-\log_{1/2}\left(1+\frac{1}{x^{1/4}}\right)-1\right] \text{ is}$$

(a) $(0, 1)$ (b) $(0, 1]$
(c) $[1, \infty)$ (d) $(1, \infty)$

**84.** If matrix $A$ is non-singular and $(A - 2I)(A - 4I) = O$, then $\left(\frac{1}{6}A+\frac{4}{3}A^{-1}\right)$ is equal to (O is null matrix)
(a) O (b) $I$ (c) $2I$ (d) $6I$

**85.** Which of the following is not a root of the equation?

$$\begin{vmatrix}x & 3 & 7\\ 2 & x & 2\\ 7 & 6 & x\end{vmatrix}=0$$

(a) $-9$ (b) 2 (c) 3 (d) 7

**86.** If $a$, $b$, $c$ are consecutive positive integers and $\log(1 + ac) = 2K$, then the value of $K$ is
(a) $\log a$ (b) $\log b$ (c) $\log c$ (d) $b$

**87.** The value of $\int_0^4 \frac{(y^2-4y+5)\sin(y-2)dy}{(2y^2-8y+1)}$ is
(a) 2 (b) 1 (c) 0 (d) $-1$

**88.** If $f(x) = x^4 \tan(x^3) - x \ln(1 + x^2)$, then the value of $\frac{d^4(f(x))}{dx^4}$ at $x = 0$ is
(a) 0 (b) 6 (c) 12 (d) 24

**89.** If $1, \omega, \omega^2, \ldots\ldots\ldots \omega^{n-1}$ are the $n^{th}$ roots of unity, then $(2 - \omega)(2 - \omega^2) \ldots\ldots (2 - \omega^{n-1})$ is equal to
(a) $2^n$ (b) $2^n - 1$
(c) $2^n + 1$ (d) 1

**90.** The locus of $Z$ satisfying the inequality $\log_{(0.8)}|Z + 1| > \log_{(0.8)}|Z - 1|$ is
(a) $\text{Im}(Z) < 0$ (b) $\text{Im}(Z) > 0$
(c) $\text{Re}(Z) < 0$ (d) $\text{Re}(Z) > 0$

**91.** If $|Z^2 - 1| = |Z|^2 + 1$, then $Z$ lies on
(a) the imaginary axis (b) the real axis
(c) $y = x$ (d) circle

**92.** If $\alpha=\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)$, then

$$\begin{vmatrix}\alpha^7 & \alpha^2 & 1\\ \alpha^8 & \alpha^6 & \alpha^4\\ \alpha^3 & \alpha & \alpha^5\end{vmatrix} \text{ is equal to}$$

(a) 0 (b) 1 (c) 2 (d) 3

**93.** If $Z_1$, $Z_2$, $Z_3$ are vertices of a triangle, then $\frac{Z_3-Z_1}{Z_3-Z_2}=i$ corresponds to a/an
(a) equilateral triangle (b) obtuse angled triangle
(c) right angled isosceles triangle
(d) ordinary triangle

**94.** If the harmonic mean between roots of $(5+\sqrt{2})x^2-bx+(8+2\sqrt{5})=0$ is 4, then $b$ is equal to
(a) 2 (b) 3
(c) $4-\sqrt{5}$ (d) $4+\sqrt{5}$

**95.** Let $\alpha$ be a repeated root of $x^3 + 3ax^2 + 3bx + c = 0$, then which of the following statements is wrong?
(a) $\alpha$ is a root of $x^2 + 2ax + b = 0$
(b) $\alpha$ is a root of $ax^2 + 2bx + c = 0$
(c) $\alpha=\frac{ab-c}{a^2-b}$ (d) $\alpha=\frac{c-ab}{2(a^2-b)}$

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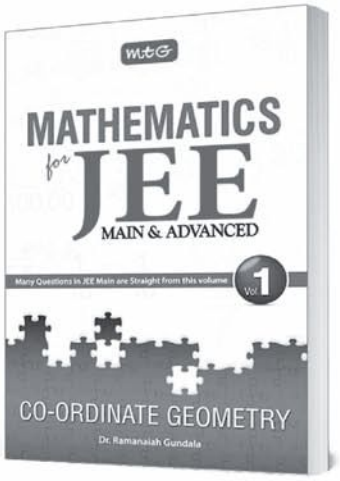


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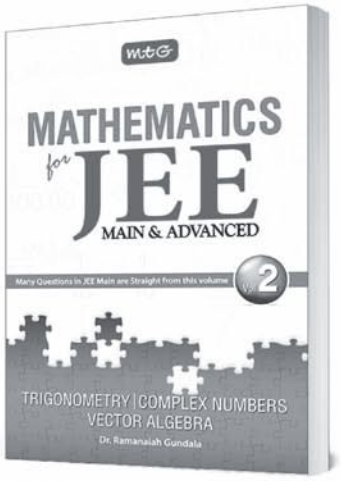


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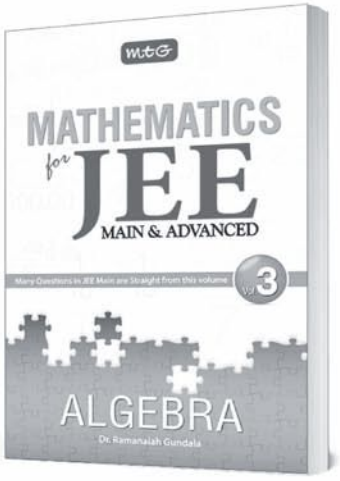


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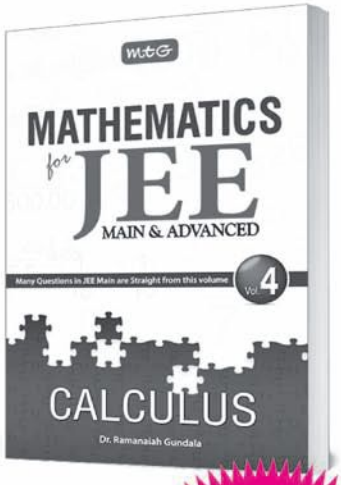


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**96.** The sum of all the real numbers satisfying the equation $x^2 + |x - 1| = 1$ is

(a) 0 (b) 1 (c) 2 (d) 4

**97.** If $a, b, c$ are three unequal numbers such that $a, b, c$ are in arithmetic progression and $b - a, c - b, a$ are in geometric progression, then ratio $a : b : c$ equal to

(a) 1 : 2 : 3 (b) 1 : 3 : 4

(c) 2 : 3 : 4 (d) 1 : 2 : 4

**98.** Sum of $n$ terms of the series $(n^2 - 1^2) + 2(n^2 - 2^2) + 3(n^2 - 3^2) + \ldots\ldots$ is

(a) $\frac{1}{4}n^2(n^2-1)$ (b) $\frac{1}{4}n(n+1)^2$

(c) 0 (d) $2n(n^2 - 1)$

**99.** The value of $\left(\frac{\log_2 24}{\log_{96} 2} - \frac{\log_2 192}{\log_{12} 2}\right)$ is

(a) 0 (b) 1 (c) 2 (d) 3

**100.** The solution set of the inequality $\log_3 (x + 2)(x + 4) + \log_{1/3} (x + 2) < \frac{1}{2} \log_{\sqrt{3}} 7$ is

(a) $(-2, -1)$ (b) $(-2, 3)$

(c) $(-1, 3)$ (d) $(3, \infty)$

## ANSWER KEYS

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | (*) | 2. | (c) | 3. | (c) | 4. | (a) | 5. | (c) |
| 6. | (c) | 7. | (a) | 8. | (d) | 9. | (b) | 10. | (d) |
| 11. | (b) | 12. | (c) | 13. | (c) | 14. | (d) | 15. | (b) |
| 16. | (c) | 17. | (b) | 18. | (d) | 19. | (a) | 20. | (d) |
| 21. | (b) | 22. | (b) | 23. | (b) | 24. | (b) | 25. | (a) |
| 26. | (c) | 27. | (b) | 28. | (d) | 29. | (b) | 30. | (b) |
| 31. | (c) | 32. | (c) | 33. | (c) | 34. | (c) | 35. | (b) |
| 36. | (a) | 37. | (a) | 38. | (c) | 39. | (a) | 40. | (c) |
| 41. | (a) | 42. | (a) | 43. | (c) | 44. | (b) | 45. | (d) |
| 46. | (*) | 47. | (a) | 48. | (c) | 49. | (d) | 50. | (b) |
| 51. | (a) | 52. | (b) | 53. | (c) | 54. | (d) | 55. | (a) |
| 56. | (d) | 57. | (d) | 58. | (b) | 59. | (*) | 60. | (b) |
| 61. | (a) | 62. | (b) | 63. | (c) | 64. | (d) | 65. | (d) |
| 66. | (b) | 67. | (a) | 68. | (d) | 69. | (c) | 70. | (b) |
| 71. | (d) | 72. | (c) | 73. | (a) | 74. | (a) | 75. | (b) |
| 76. | (b) | 77. | (c) | 78. | (d) | 79. | (a) | 80. | (c) |
| 81. | (c) | 82. | (d) | 83. | (a) | 84. | (b) | 85. | (c) |
| 86. | (b) | 87. | (c) | 88. | (a) | 89. | (b) | 90. | (c) |
| 91. | (a) | 92. | (d) | 93. | (c) | 94. | (d) | 95. | (c) |
| 96. | (b) | 97. | (a) | 98. | (a) | 99. | (d) | 100. | (b) |

*None of the option is correct


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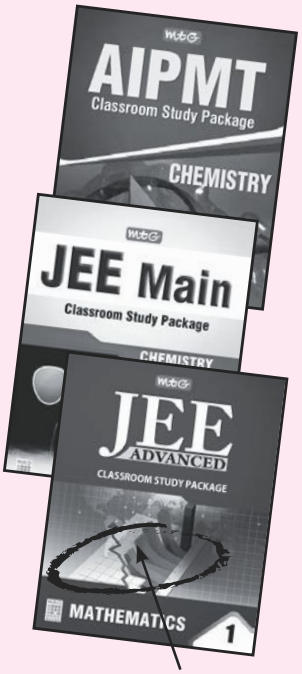


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Series 5

# ACE YOUR WAY CBSE XI

## PERMUTATIONS AND COMBINATIONS, BINOMIAL THEOREM, SEQUENCES AND SERIES

## PERMUTATIONS AND COMBINATIONS

### FUNDAMENTAL PRINCIPLE OF COUNTING

**Multiplication rule**

If a work can be done in $m$ different ways and another work can be done in $n$ different ways, then number of ways of doing the two jobs in succession is $m \times n$.

**Addition rule**

If a work can be done in $m$ different ways and another work can be done in $n$ different ways then either of the two works can be done in $(m + n)$ ways.

### FACTORIAL NOTATION

Let $n$ be a positive integer. Then, the continued product of first $n$ natural numbers is called factorial $n$ and is denoted by $n!$ or $\lfloor n$.

Thus, $n! = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$.

Also, we define $0! = 1$.

### PERMUTATIONS

The different arrangements which can be made out of a given number of objects by taking some or all at a time, are called permutations.

Let $r$ and $n$ be two positive integers such that $0 < r \leq n$. Then, the number of permutations of $n$ different objects taken $r$ at a time when repetition is not allowed is denoted by $P(n, r)$ or ${}^nP_r$ and defined by

$${}^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}.$$

***Note :***

- The number of permutations of $n$ different objects taken all at a time when repetition is not allowed is given by

$${}^nP_n = n(n-1) \ldots 2 \cdot 1 = n!$$

### IMPORTANT THEOREM

**(i)** The number of permutations of $n$ different objects taken $r$ at a time where repetition is allowed is $n^r$.

**(ii)** The number of permutations of $n$ different objects, where $p$ objects are of same kind and rest are all different is $\frac{n!}{p!}$.

Also, this theorem can be generalised as :

The number of permutations of $n$ objects, where $p_1$ objects are of one kind, $p_2$ are of second kind, ..., $p_k$ are of $k^{\text{th}}$ kind and the rest, if any, are all different is

$$\frac{n!}{p_1!\, p_2! \ldots p_k!}.$$

### COMBINATIONS

Each of the different groups or selections which can be made by taking some or all of a number of given objects at a time is called combination. In combination, order of appearance of objects is not taken into account.

Number of combinations of $n$ different objects taken $r$ at a time is denoted by ${}^nC_r$ and given by

$${}^nC_r = \frac{n!}{r!(n-r)!}, \text{ where } 0 \leq r \leq 1$$

### SOME USEFUL RESULTS ON COMBINATIONS

- ${}^nP_r = {}^nC_r\, r!,\ 0 < r \leq n$
- For $0 \leq r \leq n$, ${}^nC_r = {}^nC_{n-r}$
- For $1 \leq r \leq n$, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x = n - y$

## BINOMIAL THEOREM

If $n$ is a positive integer and $x$, $a$ are any two real numbers, then

$(x + a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}\cdot a + {}^nC_2x^{n-2}\cdot a^2 + ...$
$....+ {}^nC_rx^{n-r}\cdot a^r + ... + {}^nC_na^n$

**Observations**

- The number of terms in the binomial expansion is one more than the index $n$, *i.e.*, $n + 1$.
- In every term, sum of indices of $x$ and $a$ is $n$.
- The binomial coefficients of the terms are ${}^nC_0$, ${}^nC_1$, ${}^nC_2$,..., ${}^nC_n$.
- Since ${}^nC_r = {}^nC_{n-r}$
  $\therefore$ ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ and so on.
  Hence, the binomial coefficients of terms equidistant from the beginning and end are equal.
- The notation $\sum_{k=0}^{n} {}^nC_kx^{n-k}a^k$ stands for
  ${}^nC_0x^na^0 + {}^nC_1x^{n-1}a^1 + ... + {}^nC_rx^{n-r}a^r + ... + {}^nC_nx^0a^n$
  Hence, the theorem can also be stated as:

$$(x+a)^n = \sum_{k=0}^{n} {}^nC_kx^{n-k}a^k$$

### SOME IMPORTANT EXPANSIONS

- $(x - a)^n = {}^nC_0x^n - {}^nC_1\,x^{n-1}\,a + {}^nC_2\,x^{n-2}\,a^2 - ...$
  $...+ (-1)^n\,{}^nC_na^n$.
- $(1 + x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2x^2 + ... + {}^nC_nx^n$.
- $(1 - x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - ...+ (-1)^n\,{}^nC_n\,x^n$.

### GENERAL AND MIDDLE TERMS OF BINOMIAL EXPANSION

In the expansion of $(x + a)^n$,

| | |
|---|---|
| **General term** | $(r + 1)^{th}$ term, *i.e.*, $T_{r+1}$ is called the general term.<br>$\therefore$ General term = ${}^nC_rx^{n-r}a^r$. |
| **Middle terms** | (i) If $n$ is even, then there is only one middle term *i.e.*, $\left(\frac{n}{2}+1\right)^{th}$ term.<br>(ii) If $n$ is odd, then there are two middle terms, *i.e.*, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term. |

## SEQUENCES AND SERIES

### SEQUENCES

A succession of numbers arranged in a definite order according to a certain rule is called a sequence. A sequence is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

### SERIES

By adding the terms of a sequence, we get a series. A series is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

### ARITHMETIC PROGRESSION (A.P.)

A sequence of numbers is said to be an arithmetic progression (A.P.) if the difference between any term and the term preceding it is constant throughout. This constant is called the common difference '$d$' of the A.P.

***Note :***

- If the terms of a sequence are given be $t_1, t_2, t_3, t_4, ..., t_n$, they will be in A.P. if $t_2 - t_1 = t_3 - t_2 = ... = t_n - t_{n-1} = d$, where $d$ is the common difference.
- Let $a$ be the first term and $d$ be the common difference of an A.P., then
  General term = $n^{th}$ term *i.e.*, $t_n = a + (n - 1)d$.
- Sum to $n$ terms of an A.P. is given by

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[\text{first term} + \text{last term}]$$

### PROPERTIES OF AN ARITHMETIC PROGRESSION

**(i)** If the same quantity be added to, or subtracted from, all the terms of an A.P., the resulting progression is also an arithmetic progression.

**(ii)** If the corresponding terms of two arithmetic progressions be added or subtracted, the resulting progression is also an arithmetic progression.

**(iii)** If all the terms of an arithmetic progression be multiplied or divided by the same non-zero quantity, then the resulting progression is also an arithmetic progression.

### ARITHMETIC MEAN

**(i) A.M. between 2 numbers**

Let $a$, $b$ be the two given numbers and $A$ be the A.M. between them, then $a$, $A$, $b$ are in A.P. $\Rightarrow$ $A = \frac{a+b}{2}$

**(ii) *n* A.M.'s between two numbers**

Let $a$, $b$ be the two given numbers and $A_1, A_2, ..., A_n$ be $n$ A.M.'s between them, then

$$d = \frac{b-a}{n+1}$$

$$\therefore \quad A_1 = a + \frac{b-a}{n+1}, A_2 = a + \frac{2(b-a)}{n+1}, \ldots$$

$$\ldots, A_n = a + \frac{n(b-a)}{n+1}$$

$$\therefore \quad \text{Sum of } n \text{ A.M.'s} = A_1 + A_2 + \ldots + A_n = n\left(\frac{a+b}{2}\right)$$

**Remarks :**

**(i)** Three numbers in A.P. be taken as $a - d, a, a + d$.

**(ii)** Four numbers in A.P. be taken as $a - 3d, a - d, a + d, a + 3d$.

**(iii)** Five numbers in A.P. be taken as $a - 2d, a - d, a, a + d, a + 2d$.

## GEOMETRIC PROGRESSION (G.P.)

A succession of numbers is said to be a geometric progression (G.P.) or geometric sequence, if the ratio of any term to its preceding term is constant throughout. This constant ratio is known as the common ratio '$r$' of the G.P.

***Note :***

- A G.P. having first term $a$ and common ratio $r$ can be written as $a, ar, ar^2, \ldots, ar^{n-1}$ or $a, ar, ar^2, \ldots, ar^{n-1}, \ldots$ to $\infty$ according as the G.P. is finite or infinite respectively.
- General term = $n^{\text{th}}$ term *i.e.*, $t_n = ar^{n-1}$.
- Common ratio of a G.P. $= \left(\frac{b}{a}\right)^{\frac{1}{n-1}}$

  where $n$ = number of terms, $a$ = first term and $b$ = last term of the G.P.
- If $S_n$ be the sum of its first $n$ terms.

  Then, $S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, r<1 \\ \frac{a(r^n-1)}{r-1}, r>1 \end{cases}$

  When $r = 1$, $S_n = a + a + a + \ldots$ to $n$ terms $= na$.

## PROPERTIES OF GEOMETRIC PROGRESSION

**(i)** Three non-zero numbers $a, b, c$ are in G.P. iff $b^2 = ac$.

**(ii)** The reciprocals of the terms of a G.P. are also in G.P.

**(iii)** If each term of a G.P. be multiplied or divided by a non-zero number, then the sequence obtained is also a G.P.

**(iv)** If each term of a G.P. be raised to the same power, the resulting sequence also forms a G.P.

## GEOMETRIC MEAN

**(i) G.M. between two numbers**

Let $a$ and $b$ be any two positive numbers, then G.M. $= \sqrt{ab}$.

**(ii) *n* G.M's between two numbers**

Let $a$ and $b$ be any two positive numbers, and $G_1, G_2, \ldots, G_n$ be $n$ G.M's between them, then

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$\therefore \quad G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ldots, G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

$$\therefore \quad \text{Product of } n \text{ G.M's} = G_1 G_2 \ldots G_n = \left(\sqrt{ab}\right)^n$$

**Remarks:**

**(i)** If the three numbers are in G.P., then take the numbers as $\frac{a}{r}, a, ar$.

**(ii)** If the four numbers are in G.P., then take the numbers as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

## RELATIONSHIP BETWEEN A.M. AND G.M.

Let $A$ and $G$ be respectively A.M. and G.M. of two positive real numbers $a$ and $b$, then

$$A = \frac{a+b}{2} \quad \text{and} \quad G = \sqrt{ab}$$

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2} \geq 0 \text{ } i.e. \text{ } A \geq G$$

## SUM TO *n* TERMS OF SPECIAL SERIES

**(i)** $S_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

**(ii)** $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

**(iii)** $S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$

## Very Short Answer Type

1. How many words, with or without meaning, can be formed by using all the letters of the word 'DELHI', using each letter exactly once?
2. If ${}^nP_r = 720$ and ${}^nC_r = 120$, find $r$.
3. What is the $15^{\text{th}}$ term of the sequence defined by $t_n = \frac{1}{2n-1}$?

4. Find the number of terms in the expansion of $(1 + 2x + x^2)^{11}$.

5. Find the middle term(s) in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$.

## Short Answer Type

6. Find the term containing $x^2$, if any, in the expansion of $\left(3x - \frac{1}{2x}\right)^8$.

7. Find three numbers in G.P. whose sum is 52 and the sum of whose product in pairs is 624.

8. Find the number of numbers between 300 and 3000 which can be formed with the digits 0, 1, 2, 3, 4 and 5, no digit being repeated in any number.

9. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of
(i) exactly 3 girls (ii) atleast 3 girls.

10. If $a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$ are in A.P., then prove that $a, b, c$ are in A.P. or $ab + bc + ca = 0$.

## Long Answer Type

11. (i) If the sum of $n$ terms of an A.P. is $5n^2 + 2n$, find the series.
(ii) If the sum of $n$ terms of two A.P's are in the ratio $7n + 1 : 4n + 27$; find the ratio of their $11^{th}$ terms.

12. If the three successive coefficients in the expansion of $(1 + x)^n$ are 220, 495 and 792, then find $n$.

13. In how many ways can 3 prizes be distributed among 4 boys, when
(i) no boy get more than 1 prize
(ii) a boy may get any number of prizes
(iii) no boy get all the prizes?

14. (i) If the $n^{th}$ term of a progression is a linear expression in $n$, then show that it is an A.P.
(ii) If the $p^{th}$, $q^{th}$ and $r^{th}$ terms of an A.P. be $a, b, c$ respectively, show that
$(q - r)a + (r - p)b + (p - q)c = 0$.

15. (i) Using the binomial theorem, expand $[(x + y)^5 + (x - y)^5]$ and hence find the value of $[(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5]$.
(ii) Find the middle term(s) in the expansion of $\left(3x - \frac{x^3}{6}\right)^7$.

## SOLUTIONS

1. The word 'DELHI' contains 5 different letters.
$\therefore$ The number of required words $= {}^5P_5$
$= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

2. We know that, ${}^nC_r = \frac{{}^nP_r}{r!}$
$\therefore \quad 120 = \frac{720}{r!} \Rightarrow r! = \frac{720}{120} = 6 = 3!$
Hence, $r = 3$.

3. Substituting $n = 15$ in $t_n = \frac{1}{2n-1}$, we get
$t_{15} = \frac{1}{2 \cdot 15 - 1} = \frac{1}{29}$
$\therefore$ The $15^{th}$ term of the given sequence is $\frac{1}{29}$.

4. Given expansion $= (1 + 2x + x^2)^{11}$
$= [(1 + x)^2]^{11} = (1 + x)^{22}$
$\therefore$ The number of terms in the given expansion
$= 22 + 1 = 23$.

5. The given expansion is $\left(x - \frac{1}{x}\right)^{2n}$
There is only one middle term, *i.e.*, $T_{\frac{2n}{2}+1} = T_{n+1}$.
[$\because$ $2n$ is even]
Now, $T_{n+1} = {}^{2n}C_n (x)^n \left(-\frac{1}{x}\right)^n = (-1)^n \frac{(2n)!}{(n!)^2}$

6. Let the $(r + 1)^{th}$ term containing $x^2$ in the expansion of $\left(3x - \frac{1}{2x}\right)^8$.
$\therefore \quad T_{r+1} = {}^8C_r (3x)^{8-r} \left(-\frac{1}{2x}\right)^r$
$= {}^8C_r 3^{8-r} x^{8-r} \frac{(-1)^r}{2^r \cdot x^r}$
$= (-1)^r \frac{3^{8-r}}{2^r} {}^8C_r x^{8-2r}$ ...(i)
$\because \quad T_{r+1}$ will contain $x^2 \Rightarrow 8 - 2r = 2 \Rightarrow r = 3$.
Putting $r = 3$ in (i), we get
$T_4 = (-1)^3 \frac{3^{8-3}}{2^3} {}^8C_3 x^{8-6}$
$= -\frac{1 \cdot 3^5}{2^3} \cdot \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} x^2 = -1701x^2$
$\therefore$ The term containing $x^2$ is $T_4 = -1701$

**7.** Let the required numbers be $a$, $ar$, $ar^2$. Then,
Sum = 52 $\Rightarrow a + ar + ar^2 = 52$
$\Rightarrow a(1 + r + r^2) = 52$ ...(i)
Sum of the products in pairs = 624
$\Rightarrow a \cdot ar + ar \cdot ar^2 + a \cdot ar^2 = 624$
$\Rightarrow a^2r\,(1 + r + r^2) = 624$ ... (ii)
Dividing (ii) by (i), we get

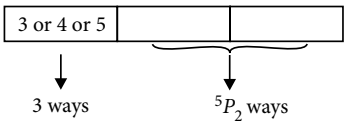
$$ar = 12 \Rightarrow a = \frac{12}{r} \quad \text{...(iii)}$$

Putting $a = \frac{12}{r}$ in (i), we get

$$\frac{12}{r}(1 + r + r^2) = 52$$

$\Rightarrow 3r^2 - 10r + 3 = 0$
$\Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = 1/3$ or $r = 3$
From (iii), $r = 3 \Rightarrow a = 4$ and $r = 1/3 \Rightarrow a = 36$.
Hence, the numbers are 4, 12, 36 or 36, 12, 4.

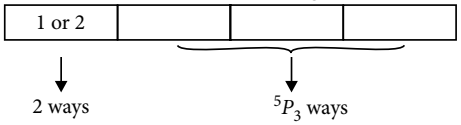
**8.** Any number between 300 and 3000 must be of 3 or 4-digits.
**Case I.** When number is of 3 digits:




Hundreds place can be filled up by any one of the three digits 3, 4 and 5 in 3 ways.
Remaining two places can be filled up by remaining five digits in $^5P_2$ ways.
$\therefore$ Number of numbers formed

$$= 3 \times {}^5P_2 = 3 \times \frac{5!}{3!} = 60$$

**Case II.** When number is of 4 digits:




$\therefore$ Number of numbers formed

$$= 2 \times {}^5P_3 = 2 \times \frac{5!}{2!} = 120$$

$\therefore$ Required number = 60 + 120 = 180

**9.** Number of boys = 9
Number of girls = 4
Number of persons to be selected = 7

| No. of girls selected | No. of boys selected | No. of ways |
|---|---|---|
| 3 | 4 | $^4C_3 \cdot {}^9C_4 = 504$ |
| 4 | 3 | $^4C_4 \cdot {}^9C_3 = 84$ |

(i) When exactly three girls are to be selected, then required number of ways = 504.
(ii) When atleast 3 girls are to be selected then required number of ways = 504 + 84 = 588.

**10.** We have, $a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$ are in A.P.
$\Rightarrow b^2(c + a) - a^2(b + c) = c^2(a + b) - b^2(c + a)$
$\Rightarrow (b^2a - a^2b) + (b^2c - a^2c)$
$= (c^2b - b^2c) + (c^2a - b^2a)$
$\Rightarrow (b - a)(ab + bc + ca) = (c - b)(ab + bc + ca)$
$\Rightarrow (2b - a - c)(ab + bc + ca) = 0$
$\Rightarrow 2b = a + c$ or $ab + bc + ca = 0$
Hence, $a$, $b$, $c$ are in A.P. or $ab + bc + ca = 0$

**11.** (i) Given, $S_n = 5n^2 + 2n$
$\therefore S_{n-1} = 5(n - 1)^2 + 2(n - 1)$
$= 5(n^2 - 2n + 1) + 2n - 2 = 5n^2 - 8n + 3$
Also, $a_n = S_n - S_{n-1} = (5n^2 + 2n) - (5n^2 - 8n + 3)$
or $a_n = 10n - 3$
Putting $n = 1, 2, 3, 4, \ldots$, the arithmetic sequence is
7, 17, 27, 37, ...
Therefore, the required arithmetic series is
7 + 17 + 27 + ....
(ii) Let $a$ be the first term and $d$ be the common difference of the first A.P. and $A$ be the first term and $D$ be the common difference of the second A.P. Then, we have

$$\frac{\frac{n}{2}[2a + (n-1)d]}{\frac{n}{2}[2A + (n-1)D]} = \frac{7n+1}{4n+27}$$

or $$\frac{2a + (n-1)d}{2A + (n-1)D} = \frac{7n+1}{4n+27} \quad \text{...(i)}$$

Now, ratio of $11^{\text{th}}$ terms $= \frac{a + 10d}{A + 10D} = \frac{2a + 20d}{2A + 20D}$
Putting $n = 21$ in (i), we get

$$\frac{2a + 20d}{2A + 20D} = \frac{147 + 1}{84 + 27} = \frac{148}{111} = \frac{4}{3}$$

$\therefore$ Required ratio = 4 : 3.

**12.** Let the successive coefficients be $^nC_{r-1}$, $^nC_r$, $^nC_{r+1}$, then, we have

$$^nC_{r-1} = 220 \quad \text{...(i)}$$
$$^nC_r = 495 \quad \text{...(ii)}$$
and $$^nC_{r+1} = 792 \quad \text{...(iii)}$$

Dividing (ii) by (i), we get

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{495}{220} \Rightarrow \frac{\lfloor n}{\lfloor n-r \rfloor r} \cdot \frac{\lfloor n-r+1 \rfloor r-1}{\lfloor n} = \frac{9}{4}$$

$$\Rightarrow \frac{n-r+1}{r} = \frac{9}{4}$$

$\Rightarrow$ $4n - 4r + 4 = 9r \Rightarrow 4n - 13r + 4 = 0$ ...(iv)

Dividing (iii) by (ii), we get

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{792}{495} \Rightarrow \frac{n-r}{r+1} = \frac{8}{5}$$

$\Rightarrow$ $5n - 5r = 8r + 8$

$\Rightarrow$ $5n - 13r - 8 = 0$ ...(v)

Solving (iv) and (v), we get $n = 12$.

**13.** (i) The first prize can be given in 4 ways as it may be given to anyone of the 4 boys.

The second prize can be given in 3 ways, because the boy who got the first prize cannot receive it.

The third prize can be given to anyone of the remaining 2 boys in 2 ways.

$\therefore$ The number of ways in which all the prizes can be distributed = $4 \times 3 \times 2 = 24$.

(ii) The first prize can be given in 4 ways as it may be given to anyone of the 4 boys.

The second prize can be given in 4 ways, because there is no restriction as to the number of prizes a student gets.

Similarly, the third prize can be given in 4 ways.

Hence, the number of ways in which all the prizes can be distributed = $4 \times 4 \times 4 = 64$.

(iii) The number of ways in which a boy gets all the 3 prizes is 4, as anyone of the 4 boys may get all the prizes.

$\therefore$ The number of ways in which a boy does not get all the prizes = $(64 - 4) = 60$.

**14.** (i) A linear expression in $n$ is of the form $an + b$, where $a$ and $b$ are constants.

Let $a_n = an + b$ ...(i)

Replacing $n$ by $(n - 1)$ in (i), we get

$a_{n-1} = a(n - 1) + b$ ...(ii)

On subtracting (ii) from (i), we get

$a_n - a_{n-1} = (an + b) - \{a(n - 1) + b\}$

$\Rightarrow$ $a_n - a_{n-1} = a$, which is a constant.

This shows that the difference between any two consecutive terms of the given progression is constant.

Hence, the given progression is an A.P.

(ii) Let $x$ be the first term and $d$ the common difference of the given A.P.

Then, $a_p = a$, $a_q = b$ and $a_r = c$

$\therefore$ $x + (p - 1)d = a$ ...(i)

$x + (q - 1)d = b$ ...(ii)

$x + (r - 1)d = c$ ...(iii)

On subtracting (ii) from (i), we get

$$(p-q)d = (a-b) \Rightarrow d = \frac{a-b}{p-q} \quad \text{... (iv)}$$

Again, subtracting (iii) from (ii), we get

$$(q-r)d = (b-c) \Rightarrow d = \frac{b-c}{q-r} \quad \text{... (v)}$$

$\therefore$ $\frac{a-b}{p-q} = \frac{b-c}{q-r}$ [from (iv) and (v)]

$\Rightarrow$ $(q - r)a - (q - r)b = (p - q)b - (p - q)c$

$\Rightarrow$ $(q - r)a + (r - p)b + (p - q)c = 0$

Hence proved.

**15.** (i) We have, $(x + y)^5 + (x - y)^5$

$= [{}^5C_0 x^5 + {}^5C_1 x^4y + {}^5C_2x^3y^2 + {}^5C_3x^2y^3 + {}^5C_4xy^4 + {}^5C_5y^5] + [{}^5C_0 x^5 - {}^5C_1 x^4y + {}^5C_2x^3y^2 - {}^5C_3x^2y^3 + {}^5C_4xy^4 - {}^5C_5y^5]$

$= 2[{}^5C_0x^5 + {}^5C_2x^3y^2 + {}^5C_4xy^4]$

$= 2[x^5 + 10x^3y^2 + 5xy^4]$

Putting $x = \sqrt{2}$ and $y = 1$, we get

$$(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5$$

$$= 2[(\sqrt{2})^5 + 10(\sqrt{2})^3 \cdot 1^2 + 5(\sqrt{2}) \cdot 1^4]$$

$$= 2[4\sqrt{2} + 20\sqrt{2} + 5\sqrt{2}] = 58\sqrt{2}.$$

(ii) Clearly, the given expansion contains 8 terms.

$\therefore$ Middle terms are $\left(\frac{8}{2}\right)^{\text{th}}$ and $\left(\frac{8}{2}+1\right)^{\text{th}}$ terms, *i.e.*, $4^{\text{th}}$ and $5^{\text{th}}$ terms.

Now, $T_4 = T_{3+1} = {}^7C_3 \cdot (3x)^4 \cdot \left(-\frac{x^3}{6}\right)^3$

$$= \left(-35 \times 81 \times x^4 \times \frac{x^9}{216}\right)$$

$$= -\frac{105x^{13}}{8}$$

And, $T_5 = T_{4+1} = {}^7C_4 \cdot (3x)^3 \cdot \left(-\frac{x^3}{6}\right)^4$

$$= \left(35 \times 27 \times x^3 \times \frac{x^{12}}{1296}\right) = \frac{35x^{15}}{48}.$$

Hence, the middle terms are $-\frac{105x^{13}}{8}$ and $\frac{35x^{15}}{48}$.

# Math Archives 10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main & Advanced) Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE (Main & Advanced). In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

**1.** The area bounded by the curve $f(x) = x + \sin x$ and its inverse function between the ordinates $x = 0$ to $x = 2\pi$ is

(a) $4\pi$ sq. units (b) $8\pi$ sq. units
(c) 4 sq. units (d) 8 sq. units

**2.** If the solution of $\frac{dy}{dx} = \frac{ax+3}{2y+f}$ represents a circle, then the value of '$a$' is

(a) 2 (b) –2 (c) 3 (d) –4

**3.** Solution of $\left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx} = \left(\frac{x+y+1}{x+y+2}\right)$, given that $y = 1$ when $x = 1$, is

(a) $\log\left|\frac{(x-y)^2-2}{2}\right| = 2(x+y)$

(b) $\log\left|\frac{(x-y)^2+2}{2}\right| = 2(x-y)$

(c) $\log\left|\frac{(x+y)^2+2}{2}\right| = 2(x-y)$

(d) None of these

**4.** Area bounded by $|x - 1| \le 2$ and $x^2 - y^2 = 1$ is

(a) $6\sqrt{2} + \frac{1}{2}\log|3+2\sqrt{2}|$ (b) $6\sqrt{2} + \frac{1}{2}\log|3-2\sqrt{2}|$

(c) $6\sqrt{2} - \log|3+2\sqrt{2}|$ (d) None of these

**5.** Area bounded by $y = \frac{(x-1)(x+1)}{(x-2)}$, $x$-axis and ordinates $x = 0$ and $x = 3/2$ is

(a) 4/5 (b) 7/8
(c) 1 (d) None of these

**6.** Evaluate: $\int e^x \frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\,dx$

**7.** Evaluate: $\int \frac{(x\cos x+1)}{\sqrt{2x^3e^{\sin x}+x^2}}\,dx$

**8.** Evaluate: $\int \frac{(2x-1)\,dx}{x^4-2x^3+x+1}$

**9.** Evaluate : $\int_0^1 \frac{x^{\cos a}-1}{\ln x}dx$, '$a$' being a real number other than an odd multiple of $\pi$.

**10.** Evaluate: $\int \frac{\cos 6x+\cos 9x}{1-2\cos 5x}dx$

## SOLUTIONS

**1. (d):** Required area = $4A$

$$A = \int_0^{\pi}(x+\sin x)\,dx - \int_0^{\pi} x\,dx$$

$$= \frac{\pi^2}{2} - \cos\pi + \cos 0 - \frac{\pi^2}{2} = 2 \text{ sq. units.}$$

**2. (b):** We have,

$$\frac{dy}{dx} = \frac{ax+3}{2y+f} \Rightarrow (2y+f)\,dy = (ax+3)\,dx$$

$$\Rightarrow \frac{2y^2}{2} + fy = \frac{ax^2}{2} + 3x + c$$

$\therefore$ For curve to be circle, $a = -2$ where $f^2 + 9 + 4c > 0$.

**3. (d):** We have, $\left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx} = \left(\frac{x+y+1}{x+y+2}\right)$

Put $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$$\therefore \quad \frac{dt}{dx} - 1 = \left(\frac{t+1}{t+2}\right)\left(\frac{t-2}{t-1}\right) \Rightarrow \frac{dt}{dx} = \left(\frac{t^2-t-2}{t^2+t-2}\right) + 1$$

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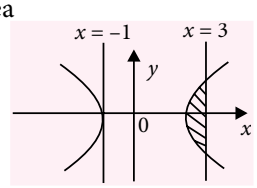
On solving, we get

$$\Rightarrow \quad 2(y-x)+\log\left(\frac{(x+y)^2-2}{2}\right)=0$$

**4. (c) :** Graph of given curve $-1 \le x \le 3$ and $x^2 - y^2 = 1$ is shown below

$\therefore$ Required area is shaded area




*i.e.*, $2\int_1^3 \sqrt{x^2-1}\,dx$

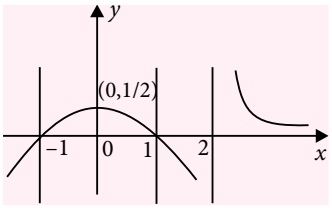
$= 6\sqrt{2} - \log|3+2\sqrt{2}|$

**5. (b) :** Graph of given curve is shown below.

$\therefore$ Required area is

$$\int_0^1 \frac{(x-1)(x+1)}{(x-2)}dx - \int_1^{3/2} \frac{(x-1)(x+1)}{(x-2)}dx$$

Hence area = 7/8




**6.** Let $I = \int e^x \frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}dx$

$$= \int e^x \left\{\frac{(1-x^{2n})}{(1-x^n)\sqrt{1-x^{2n}}} + \frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right\}dx$$

$$= \int e^x \left\{\frac{\sqrt{1-x^{2n}}}{1-x^n} + \frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right\}dx$$

$$= e^x \cdot \frac{\sqrt{1-x^{2n}}}{1-x^n} + c \quad \left[\because \int e^x(f(x)+f'(x))dx = e^x f(x)+c\right]$$

**7.** Let $I = \int \frac{(x\cos x+1)}{\sqrt{2x^3e^{\sin x}+x^2}}dx = \int \frac{(x\cos x+1)}{x\sqrt{2xe^{\sin x}+1}}dx$

Put $2xe^{\sin x} + 1 = t^2$

$\Rightarrow$ $(2xe^{\sin x} \cdot \cos x + 2e^{\sin x})\,dx = 2t\,dt$

$\Rightarrow$ $e^{\sin x}(x\cos x + 1)\,dx = t\,dt$

$$\therefore \quad I = \int \frac{t\,dt}{xe^{\sin x}.t} = \int \frac{dt}{xe^{\sin x}} = \int \frac{2dt}{t^2-1}$$

$$= \int \frac{(1+t)+(1-t)}{(t^2-1)}dt = \int \frac{dt}{t-1} - \int \frac{dt}{t+1}$$

$$= \log(t-1) - \log(t+1) + c$$

$$= \log\left(\frac{t-1}{t+1}\right) + c \text{ where } t^2 = 2xe^{\sin x} + 1$$

**8.** $I = \int \frac{(2x-1)dx}{x^4-2x^3+x+1} = \int \frac{(2x-1)dx}{(x^2-x)^2-(x^2-x)+1}$

Put $x^2 - x = t \Rightarrow (2x - 1)dx = dt$

$$\therefore \quad I = \int \frac{dt}{t^2-t+1} = \int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)+c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2(x^2-x)-1}{\sqrt{3}}\right]+c$$

**9.** $\int_0^1 \frac{x^{\cos a}-1}{\ln x}dx$ is a function of '$a$', so let us call it $I(a)$.

Then $I(a) = \int_0^1 \frac{x^{\cos a}-1}{\ln x}dx$

Differentiating w.r.t. '$a$', we get

$$I'(a) = \int_0^1 \frac{x^{\cos a}.\ln x\,(-\sin a)}{\ln x}dx \quad \left(\because \frac{d}{dx}(b^x) = b^x \ln b\right)$$

$$= -\int_0^1 x^{\cos a}\cdot \sin a\,dx = -\sin a\int_0^1 x^{\cos a}dx$$

$$= -\sin a\left[\frac{x^{\cos a+1}}{\cos a+1}\right]_0^1 = -\sin a\left[\frac{1}{\cos a+1}\right]$$

$$\therefore \quad I'(a) = \frac{-\sin a}{\cos a+1}$$

Integrating both sides w.r.t. '$a$', we get

$$I(a) = \int \frac{-\sin a\,da}{\cos a+1} = \int \frac{d(\cos a+1)}{\cos a+1}$$

or $I(a) = \ln|1 + \cos a| + c$.

As $a \to \frac{\pi}{2}$, $I(a) \to 0 \Rightarrow c = 0$

Thus $I(a) = \ln(1 + \cos a)$, where $a \neq$ odd multiple of $\pi$.

**10.** Let $I = \int \frac{\cos 6x+\cos 9x}{1-2\left(2\cos^2\frac{5x}{2}-1\right)}dx$

$$= \int \frac{2\cos\frac{15x}{2}.\cos\frac{3x}{2}}{3-4\cos^2\frac{5x}{2}}dx = \int \frac{2\cos\frac{15x}{2}.\cos\frac{3x}{2}.\cos\frac{5x}{2}}{3\cos\frac{5x}{2}-4\cos^3\frac{5x}{2}}dx$$

$$= -\int(\cos 4x+\cos x)dx = -\frac{\sin 4x}{4} - \sin x + C$$

# CONCEPT BOOSTERS

Class XI

## TRIGONOMETRIC RATIOS AND IDENTITIES

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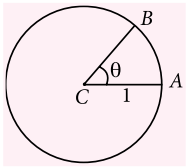
This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

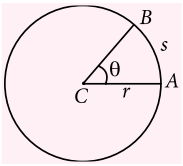
### RADIAN MEASURE

The radian measure θ of angle *ACB* is defined to belength of the circular arc *AB* of unit radius circle, Since the circumference of the circle is 2π and one complete revolution of a circle is 360°.




The relation between radians and degrees is given by the following equation π radian = 180°

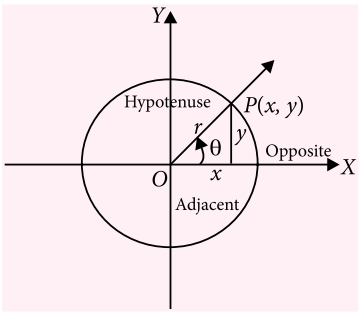
**Radian Measure and Arc length :** There is a useful relationship between the length of an arc *AB* on a circle of radius *r* and the radian measure θ is the angle that the arc subtends at the circle's centre *C*.




Here $\frac{s}{r} = \theta$ or $s = r\theta$

### THE SIX BASIC TRIGONOMETRIC FUNCTIONS

We define the trigonometric functions in terms of the co-ordinates of the point $P(x, y)$, where the angle's terminal ray intersects the circle.




$$\text{sine} : \sin\theta = \frac{y}{r} \qquad \text{cosecant} : \operatorname{cosec}\theta = \frac{r}{y}$$

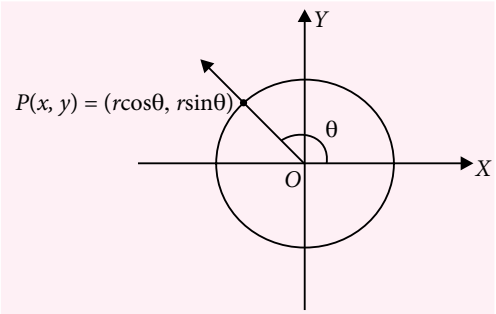
$$\text{cosine} : \cos\theta = \frac{x}{r} \qquad \text{secant} : \sec\theta = \frac{r}{x}$$

$$\text{tangent} : \tan\theta = \frac{y}{x} \qquad \text{cotangent} : \cot\theta = \frac{x}{y}$$

Here, tanθ and secθ are not defined if $x = 0$, This means they are not defined if θ is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\ldots$ and similarly cotθ and cosecθ are not defined for values of θ for which $y = 0$,

*i.e.* $\theta = 0, \pm\pi, \pm 2\pi, \ldots\ldots$

The co-ordinate of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance from the origin and the angle that ray *OP* makes with the positive *x*-axis.




We have, $x = r\cos\theta, \quad y = r\sin\theta$

or $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$

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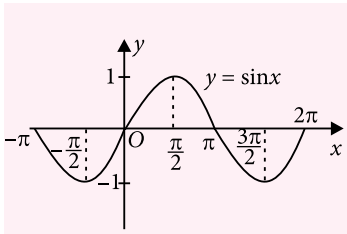
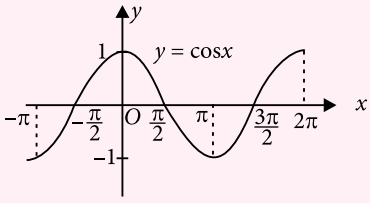
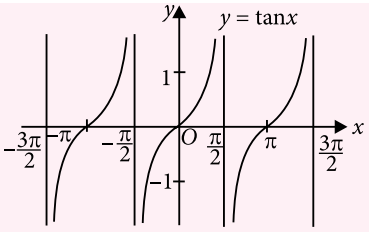
**Graphs :** When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by $x$ instead of θ, the variations in the values of the trigonometric ratios may be graphically represented in the following manner.



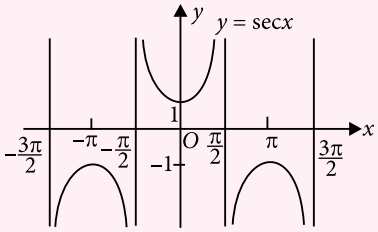

Domain : $(-\infty, \infty)$
Range : $[-1, 1]$



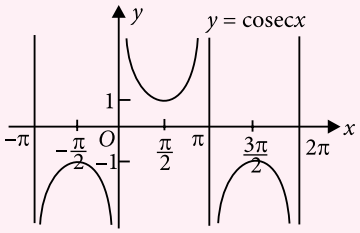

Domain : $(-\infty, \infty)$
Range : $[-1, 1]$



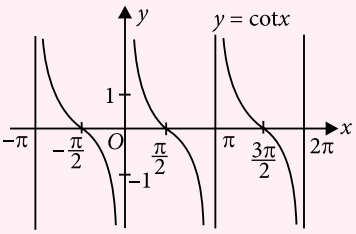

Domain : All real numbers except odd integer multiples of $\frac{\pi}{2}$
Range: $(-\infty, \infty)$




Domain : $(-\infty, \infty) - \{x : x = (2n + 1)\frac{\pi}{2}, n \in Z\}$
Range: $(-\infty, -1] \cup [1, \infty)$




Domain : $(-\infty, \infty) - \{x : x = n\pi, n \in Z\}$
Range: $(-\infty, -1] \cup [1, \infty)$




Domain : $(-\infty, \infty) - \{x : x = n\pi, n \in Z\}$
Range: $(-\infty, \infty)$

## Signs of Trigonometric Ratios:

| B | |
|---|---|
| In the second quadrant, the | In the first quadrant, the |
| Sine decreases from 1 to 0 | Sine increases from 0 to 1 |
| cosine decreases from 0 to −1 | cosine decreases from 1 to 0 |
| tangent increases from −∞ to 0 | tangent increases from 0 to ∞ |
| cotangent decreases from 0 to −∞ | cotangent decreases from ∞ to 0 |
| secant increases from −∞ to −1 | secant increases from 1 to ∞ |
| cosecant increases from 1 to ∞ | cosecant decreases from ∞ to 1 |
| A′ O | A |
| In the third quadrant, the | In the fourth quadrant, the |
| Sine decreases from 0 to 1 | Sine increases from −1 to 0 |
| cosine decreases from −1 to 0 | cosine increases from 0 to 1 |
| tangent increases from 0 to ∞ | tangent increases from −∞ to 0 |
| cotangent decreases from 0 to ∞ | cotangent decreases from 0 to −∞ |
| secant increases from −1 to −∞ | secant decreases from ∞ to 1 |
| cosecant increases from −∞ to −1 | cosecant decreases from −1 to −∞ |
| B′ | |

**Periods of the Trigonometrical Functions :** As an angle increases from 0 to 2π radians. *i.e.*, while the revolving line makes a complete revolution, its sine first increases from 0 to 1, then decreases from 1 to −1, and finally increases from −1 to 0, and thus the sine goes through all its changes, returning to its original value.

Similarly, as the angle increases from 2π radians to 4π radians, the sine goes through the same series of same changes.

Also, the sines of any two angles which differ by four right angles, *i.e.*, 2π radians, are same.

This is expressed by saying that the period of the sine function is 2π.

Similarly, the cosine, secant, and cosecant go through all their changes as the angle increases by 2π.

The tangent, however, goes through all its changes as the angle increases from 0 to π radians, *i.e.*, while the revolving line turns through two right angles. Similarly for the cotangent.

The period of the sine, cosine, secant and cosecant is therefore 2π radians; the period of the tangent and cotangent is π radians.

Since the values of the trignometrical functions repeat over and over again as the angle increases, they are called periodic functions.

**PERIOD :** The positive value of "$a$" such that $f(x + a) = f(x)$ then "$a$" is called period of $f(x)$.

## TRIGONOMETRIC RATIOS FOR AN ANGLE OF ANY MAGNITUDE

**Complementary Angles:**

Two angles are said to be complementary when their sum is equal to a right angle. Thus any angle $\theta$ and the angle $90° - \theta$ are complementary.

**Supplementary angles:**

Two angles are said to be supplementary when their sum is equal to two right angles, *i.e.*, the supplement of any angle $\theta$ is $180° - \theta$.

## TRIGONOMETRIC RATIOS OF COMPOUND ANGLES

An angle made up of the algebraic sum of two or more angles is called compound angle.

Some formulae and results regarding compound angles:

- $\sin(A + B) = \sin A\cos B + \cos A\sin B$
- $\sin(A - B) = \sin A\cos B - \cos A\sin B$
- $\cos(A + B) = \cos A\cos B - \sin A\sin B$
- $\cos(A - B) = \cos A\cos B + \sin A\sin B$
- $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$,

  $\tan(45° + A) = \dfrac{1+\tan A}{1-\tan A}$
- $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$,

  $\tan(45° - A) = \dfrac{1-\tan A}{1+\tan A}$
- $\cot(A + B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$,

  $\cot(A - B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$
- $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$
  $= \cos^2 B - \cos^2 A$
- $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$
  $= \cos^2 B - \sin^2 A$
- $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

## TRIGONOMETRIC RATIOS OF MULTIPLES OF AN ANGLE:

- $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$
  $= \dfrac{1-\tan^2 A}{1+\tan^2 A}$

  $1 + \cos 2A = 2\cos^2 A$, $1 - \cos 2A = 2\sin^2 A$
- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
  $= 4\sin(60° - A)\sin A\sin(60° + A)$
- $\cos 3A = 4\cos^3 A - 3\cos A$
  $= 4\cos(60° - A)\cos A\cos(60° + A)$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
  $= \tan(60° - A)\tan A\tan(60° + A)$

## PRODUCT OF SINES/COSINES IN TERM OF SUMS

- $2\sin A\cos B = \sin(A + B) + \sin(A - B)$
- $2\cos A\sin B = \sin(A + B) - \sin(A - B)$
- $2\cos A\cos B = \cos(A + B) + \cos(A - B)$
- $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

## SUM OF SINES/ COSINES IN TERM OF PRODUCT

- $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
- $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$
- $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
- $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$
- $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A\cos B}$,

  $\tan A - \tan B = \dfrac{\sin(A-B)}{\cos A\cos B}$

## MAXIMUM AND MINIMUM VALUES OF $f(\theta) = a\cos\theta + b\sin\theta$, $\theta \in R$

Let $a = r\sin\alpha$, $b = r\cos\alpha$ so that

$r = \sqrt{a^2 + b^2}$, $r \in R$

Also, $a\cos\theta + b\sin\theta = r(\cos\theta\sin\alpha + \sin\theta\cos\alpha)$
$= r\sin(\theta + \alpha)$

Now the maximum and minimum values of $\sin(\theta + \alpha)$ are 1 & –1 respectively.

Hence, $-r \le \sin(\theta + \alpha) \le r$

$\Rightarrow \quad -\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$

Hence, the maximum value is $\sqrt{a^2+b^2}$ and minimum value is $-\sqrt{a^2+b^2}$.

## IDENTITIES

A trigonometric equation is an identity if it is true for all values of the angle or angles involved. A given identity may be established by (i) reducing either side to the other one, or (ii) reducing each side to the same expression, or (iii) any convenient modification of the methods given in (i) & (ii).

## CONDITIONAL IDENTITIES

When the angles $A$, $B$ and $C$ satisfy a given relation, many interesting identities can be established connecting the trigonometric functions of these angles. In providing these identities, we require the properties of complementary and supplementary angles. For example, if $A + B + C = \pi$, then

- $\sin(B + C) = \sin A$, $\cos B = -\cos(C + A)$
- $\cos(A + B) = -\cos C$, $\sin C = \sin(A + B)$
- $\tan(C + A) = -\tan B$, $\cot A = -\cot(B + C)$
- $\cos\frac{A+B}{2} = \sin\frac{C}{2}$, $\cos\frac{C}{2} = \sin\frac{A+B}{2}$
- $\sin\frac{C+A}{2} = \cos\frac{B}{2}$, $\sin\frac{A}{2} = \cos\frac{B+C}{2}$
- $\tan\frac{B+C}{2} = \cot\frac{A}{2}$, $\tan\frac{B}{2} = \cot\frac{C+A}{2}$

## SOME IMPORTANT IDENTITIES:

If $A$, $B$, $C$ are angles of a triangle (or $A + B + C = \pi$), then

- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
- $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$
- $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$
- $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$
- $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$
- $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$
- $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
- $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

## TWO SIMPLE TRIGONOMETRICAL SERIES

- $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin\{\alpha+(n-1)\beta\} = \dfrac{\sin\left\{\frac{2\alpha+(n-1)\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$
- $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha+(n-1)\beta\} = \dfrac{\cos\left\{\frac{2\alpha+(n-1)\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$

## TRIGONOMETRIC EQUATIONS

An equation involving one or more trigonometrical ratios of unknown angle is called trigonometric equation *e.g.* $\cos^2 x - 4\sin x = 1$. It is to be noted that a trigonometrical identity is satisfied for every value of the unknown angle whereas, trigonometric equation is satisfied only for some values (finite or infinite in number) of unknown angle. *e.g.* $\sin^2 x + \cos^2 x = 1$ is a trigonometrical identity as it is satisfied for every value of $x \in R$.

## SOLUTION OF A TRIGONOMETRIC EQUATION

A value of the unknown angle which satisfies the given equation is called a solution of the equation *e.g.* $\theta = \pi/6$ is a solution of $\sin\theta = \frac{1}{2}$.

## GENERAL SOLUTION

Since trigonometrical functions are periodic functions, solutions of trigonometric equations can be generalized with the help of the periodicity of the trigonometrical functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

We use the following formulae for solving the trigonometric equations:

If $\sin\theta = \sin\alpha$ then $\theta = n\pi + (-1)^n\alpha$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

## TYPES OF TRIGONOMETRIC EQUATIONS :

- Solution of equations by factorising.
- Solutions of equations reducible to quadratic equations.
- Solving equations by introducing an Auxiliary argument.
- Solving equations by transforming a sum of Trigonometric functions into a product.

- Solving equations by transforming a product of trigonometric functions into a sum.
- Solving equations by a change of variable :
- Equations of the form
  $P(\sin x \pm \cos x, \sin x \cos x) = 0$, can be solved by the change
  $\cos x \pm \sin x = t \Rightarrow 1 \pm 2\sin x \cos x = t^2$.
  Consider the equation;
  $\sin x + \cos x = 1 + \sin x \cos x$ .
- Equation of the form of $a\sin x + b\cos x + d = 0$, where $a$, $b$ and $d$ are real numbers and $a$, $b \neq 0$ can be solved by changing $\sin x$ and $\cos x$ into their corresponding tangent of half the angle. Consider the equation $3\cos x + 4\sin x = 5$.
- Many equations can be solved by introducing a new variable *e.g.* the equation $\sin^4 2x + \cos^4 2x = \sin 2x \cos 2x$ changes to $2(y+1)\left(y-\frac{1}{2}\right)=0$ by substituting, $\sin 2x \cos 2x = y$.
- Solving equations with the use of the Boundness of the functions $\sin x$ and $\cos x$. Consider the equation;

$$\sin x\left(\cos\frac{x}{4}-2\sin x\right)+\left(1+\sin\frac{x}{4}-2\cos x\right)\cos x=0$$

## TRIGONOMETRIC INEQUATIONS

To solve trigonometric inequation of the type $f(x) \leq a$, or $f(x) \geq a$ where $f(x)$ is some trigonometric ratio, we take following steps.

(i) Draw the graph of $f(x)$ in a interval length equal to fundamental period of $f(x)$.
(ii) Draw the line $y = a$.
(iii) Take the portion of the graph for which inequation is satisfied.
(iv) To generalise add $pn$ $(n \in I)$ and take union over set of integers, where $p$ is fundamental period of $f(x)$.

## PROBLEMS BASED ON BOUNDARY CONDITIONS

If the problem involves only one equation consisting of more than one variable or equation involves variable of different nature then the boundary conditions of trigonometric functions is generally used. It must be noted that $|\sin x| \leq 1$; $|\cos x| \leq 1$; $|\sec x| \geq 1$; $|\operatorname{cosec} x| \geq 1$; $|\tan x| \geq 0$; $|\cot x| \geq 0$

# PROBLEMS

## SECTION-I

### Single Correct Answer Type

**1.** If $\tan\left(\frac{\pi}{4}+\frac{y}{2}\right)=\tan^3\left(\frac{\pi}{4}+\frac{x}{2}\right)$, then $\sin x\left(\frac{3+\sin^2 x}{1+3\sin^2 x}\right)$ equals
(a) $\cos y$ (b) $\sin y$ (c) $\sin 2y$ (d) 0

**2.** The number of solutions of the equation $16(\sin^5 x + \cos^5 x) = 11(\sin x + \cos x)$ in the interval $[0, 2\pi]$ is
(a) 6 (b) 7 (c) 8 (d) 9

**3.** Sum of integral values of $n$ such that $\sin x(2\sin x + \cos x) = n$, has at least one real solution is
(a) 3 (b) 1 (c) 2 (d) 0

**4.** The equation $2x = (2n + 1)\pi (1 - \cos x)$, (where $n$ is a positive integer)
(a) has infinitely many real roots
(b) has exactly one real root
(c) has exactly $2n + 2$ real roots
(d) has exactly $2n + 3$ real roots

**5.** If $\sin A = \sin B$ and $\cos A = \cos B$, $A > B$, then
(a) $\sin (1/2) (A - B) = 0$ (b) $\sin (1/2) (A + B) = 0$
(c) $\cos (1/2) (A - B) = 0$ (d) $\cos (1/2) (A + B) = 0$

**6.** If $\log_{0.5} \sin x = 1 - \log_{0.5} \cos x$ then number of values of $x \in [-2\pi, 2\pi]$ is
(a) 1 (b) 2 (c) 3 (d) 4

**7.** If $2^{\sqrt{\sin^2 x - 2\sin x + 5}} \frac{1}{4^{\sin^2 y}} \leq 1$, then the ordered pair $(x, y)$ is equal to $(m, n \in I)$
(a) $x=(4n+1)\frac{\pi}{2},\ y=(2m+1)\frac{\pi}{2}$
(b) $x = 2n\pi, y = 2m\pi$
(c) $x=(2n+1)\frac{\pi}{2},\ y=(2m+1)\frac{\pi}{2}$
(d) $x = n\pi, y = m\pi$

**8.** A set of values of $x$, satisfying the equation $\cos^2\left(\frac{1}{2}px\right)+\cos^2\left(\frac{1}{2}qx\right)=1$ form an arithmetic progression with common difference
(a) $\frac{2}{p+q}$ (b) $\frac{2}{p-q}$
(c) $\frac{\pi}{p+q}$ (d) none of these

**9.** The value of $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50° =$

(a) $\frac{4}{5}$ (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) 3

**10.** The least positive values of $x$ satisfy the equation $8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\ldots\infty} = 4^3$ will be (where $|\cos x| < 1$)

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$

(c) $\frac{\pi}{4}$ (d) none of these

**11.** Value of $\frac{3+\cot 80° \cot 20°}{\cot 80° + \cot 20°}$ is equal to

(a) cos50° (b) tan50°

(c) sin50° (d) none of these

**12.** If $\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$, then maximum value of $\Delta$ is

(a) 1 (b) 9

(c) 16 (d) none of these

**13.** Number of ordered pairs $(a, x)$ satisfying the equation $\sec^2(a+2)x + a^2 - 1 = 0$; $-\pi < x < \pi$ is

(a) 2 (b) 4

(c) 3 (d) none of these

**14.** If $a, b, c$ be the sides of a triangle $ABC$ and the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, then $\sin^2\left(\frac{A}{2}\right)$, $\sin^2\left(\frac{B}{2}\right)$, $\sin^2\left(\frac{C}{2}\right)$ are in

(a) A.P. (b) G.P. (c) A.G.P. (d) H.P.

**15.** In $\Delta ABC$, if $\cos A + \sin A - \frac{2}{\cos B + \sin B} = 0$, then $\frac{a+b}{c}$ is equal to

(a) $\sqrt{2}$ (b) 1 (c) $\frac{1}{\sqrt{2}}$ (d) $2\sqrt{2}$

**16.** In a triangle $ABC$, medians $AD$ and $CE$ are drawn. If $AD = 5$, $\angle DAC = \frac{\pi}{8}$ and $\angle ACE = \frac{\pi}{4}$ then the area of triangle $ABC$ is equal to

(a) $\frac{25}{9}$ (b) $\frac{25}{3}$ (c) $\frac{25}{18}$ (d) $\frac{10}{3}$

**17.** In a right angled triangle $ABC$ with $A = \frac{\pi}{2}$, a circle is drawn touching the side $AB$, $AC$ and incircle of the triangle. It's radius is equal to

(a) $(2-\sqrt{2})r$ (b) $(3-\sqrt{2})r$

(c) $(3+\sqrt{2})r$ (d) $(3-2\sqrt{2})r$

**18.** Three positive real numbers $x, y, z$ satisfy the equations $x^2 + \sqrt{3}xy + y^2 = 25$, $y^2 + z^2 = 9$ and $x^2 + xz + z^2 = 16$ then the value of $xy + 2yz + \sqrt{3}xz$ is

(a) 18 (b) 24 (c) 30 (d) 36

**19.** The sides of a triangle inscribed in a given circle subtend angles $\alpha$, $\beta$, $\gamma$ at the centre. Then, the minimum value of the A.M of $\cos\left(\alpha + \frac{\pi}{2}\right), \cos\left(\beta + \frac{\pi}{2}\right), \cos\left(\gamma + \frac{\pi}{2}\right)$ is

(a) $-\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) none of these

**20.** In a scalene $\Delta ABC$, $D$ is a point on the side $AB$ such that $CD^2 = AD \cdot DB$, if $\sin A \cdot \sin B = \sin^2\frac{C}{2}$ then $CD$ is

(a) Median through $C$ (b) Internal bisector of $\angle C$

(c) Altitude through $C$

(d) Divide $AB$ in the ratio 1 : 2

**21.** The radii of the escribed circles of $\Delta ABC$ are $r_a$, $r_b$ and $r_c$ respectively. If $r_a + r_b = 3R$ and $r_b + r_c = 2R$, then the smallest angle of triangle is

(a) $\tan^{-1}(\sqrt{2}-1)$ (b) $\frac{1}{2}\tan^{-1}(\sqrt{3})$

(c) $\frac{1}{2}\tan^{-1}(\sqrt{2}+1)$ (d) $\tan^{-1}(2-\sqrt{3})$

**22.** With usual notations, in a triangle $ABC$, $a\cos(B-C) + b\cos(C-A) + c\cos(A-B)$ is equal to

(a) $\frac{abc}{R^2}$ (b) $\frac{abc}{4R^2}$

(c) $\frac{4abc}{R^2}$ (d) $\frac{abc}{2R^2}$

**23.** If in a triangle $ABC$, $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3c}{2}$ then minimum value of $\frac{a+c}{2c-a} + \frac{b+c}{2c-b}$ is equal to

(a) 2 (b) 4 (c) 6 (d) 8

**24.** In an isosceles triangle $ABC$, $AB = AC$. If vertical angle $A$ is 20°, then $a^3 + b^3$ is equal to

(a) $3a^2b$ (b) $3b^2c$ (c) $3c^2a$ (d) $abc$

**25.** In acute angled triangle $ABC$, $r + r_1 = r_2 + r_3$ and $\angle B > \frac{\pi}{3}$ then

(a) $b + 2c < 2a < 2b + 2c$
(b) $b + 4c < 4a < 2b + 4c$
(c) $b + 4c < 4a < 4b + 4c$
(d) $b + 3c < 3a < 3b + 3c$

**26.** If $\cot\frac{A}{2} = \frac{b+c}{a}$, then $\Delta ABC$ is

(a) isosceles (b) equilateral
(c) right angled (d) none of these

**27.** In a triangle $ABC$, $(a + b + c)(b + c - a) = \lambda bc$ if

(a) $\lambda < 0$ (b) $\lambda > 6$
(c) $0 < \lambda < 4$ (d) $\lambda > 4$

## SECTION-II

### Multiple Correct Answer Type

**28.** If $(1 + k)\tan^2 x - 4\tan x - 1 + k = 0$ has real roots $\tan x_1$ and $\tan x_2$, then

(a) $k^2 \le 5$ (b) $\tan(x_1 + x_2) = 2$
(c) for $k = 2$, $x_1 = \frac{\pi}{4}$ (d) for $k = 1$, $x_1 = 0$

**29.** The solution set of $|\sin x| \le |\cos 2x|$ contains

(a) $\bigcup_{n\in I}\left\{\left[n\pi - \frac{\pi}{6}, n\pi + \frac{\pi}{6}\right]\right\}$

(b) $\bigcup_{n\in I}\left\{n\pi + \frac{\pi}{2}\right\}$

(c) $\bigcup_{n\in I}\left\{\left[n\pi - \frac{\pi}{8}, n\pi + \frac{\pi}{8}\right]\right\}$

(d) $\bigcup_{n\in I}\left\{\left[n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{4}\right]\right\}$

**30.** For $0 < \theta < \frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$ is/are

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$

**31.** $\sin\theta + \sqrt{3}\cos\theta = 6x - x^2 - 11$, $0 \le \theta \le 4\pi$, $x \in R$ holds for

(a) no value of $x$ and $\theta$
(b) one value of $x$ and two values of $\theta$
(c) two values of $x$ and two values of $\theta$
(d) two pairs of values of $(x, \theta)$

**32.** The solutions of the system of equations

$\sin x \sin y = \frac{\sqrt{3}}{4}$, $\cos x \cos y = \frac{\sqrt{3}}{4}$ are

(a) $x_1 = \frac{\pi}{3} + \frac{\pi}{2}(2n + k); n, k \in I$

(b) $y_1 = \frac{\pi}{6} + \frac{\pi}{2}(k - 2n); n, k \in I$

(c) $x_2 = \frac{\pi}{6} + \frac{\pi}{2}(2n + k); n, k \in I$

(d) $y_2 = \frac{\pi}{3} + \frac{\pi}{2}(k - 2n); n, k \in I$

**33.** Values of $x \in (-\pi, \pi)$ satisfying the equation

$(\sqrt{3}\sin x + \cos x)^{\sqrt{\sqrt{3}\sin 2x - \cos 2x + 2}} = 4$ are,

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
(c) $-\frac{\pi}{3}$ (d) $-\frac{2\pi}{3}$

**34.** Which of following functions have the maximum value unity?

(a) $\sin^2 x - \cos^2 x$

(b) $\sqrt{\frac{6}{5}}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{3}}\cos x\right)$

(c) $\cos^6 x + \sin^6 x$ (d) $\cos^2 x + \sin^4 x$

**35.** In a $\Delta AEX$, $T$ is the mid point of $XE$, and $P$ is the mid point of $ET$. If the $\Delta APE$ is equilateral of side length equal to unity then which of the following alternative(s) is/are correct?

(a) $AX = \sqrt{13}$ (b) $\angle EAT = 90°$
(c) $\cos\angle XAE = \frac{-1}{\sqrt{13}}$ (d) $AT = \frac{1}{\sqrt{3}}$

**36.** In $\Delta ABC$, $\angle C = 2\angle A$ and $AC = 2BC$, then which of the following is/are true?

(a) Angles $A$, $B$, $C$ are in arithmetic progression.
(b) Angles $A$, $C$, $B$ are in arithmetic progression.
(c) $\Delta ABC$ is a right angled isosceles triangle.
(d) $BC^2 + CA^2 + AB^2 = 8R^2$, where $R$ is the circumradius of $\Delta ABC$.

**37.** If in a triangle $ABC$, $BC = 5$, $CA = 4$, $AB = 3$ and $D, E$ are points on $BC$ such that $BD = DE = EC$, $\angle CAE = \theta$ then

(a) $AE^2 = 73/3$ (b) $AE^2 = 73/9$
(c) $\tan\theta = 3/8$ (d) $\cos\theta = \frac{3}{\sqrt{73}}$

**38.** If $A$ is the area and $2S$ the sum of sides of a triangle, then

(a) $A < \frac{S^2}{4}$ (b) $A \le \frac{S^2}{3\sqrt{3}}$

(c) $A > \frac{S^2}{\sqrt{3}}$ (d) $A \le \frac{S^2}{\sqrt{3}}$

## SECTION-III

### Comprehension Type

**Paragraph for Question No. 39 & 40**

If the curves $y = f(x)$ and $y = g(x)$ intersects at $n$ different points then $f(x) = g(x)$ is said to have '$n$' solutions.

**39.** Number of solutions of $|\cos x| = 2[x]$ (where [ ] is step function) is

(a) 0 (b) 1

(c) 2 (d) Infinite

**40.** The number of solutions of $\sin \pi x = |\log_e |x||$ is

(a) 0 (b) 6 (c) 4 (d) 8

**Paragraph for Question No. 41 to 43**

Given $\cos 2^m\theta \cos 2^{m+1}\theta \ldots \cos 2^n\theta = \frac{\sin 2^{n+1}\theta}{2^{n-m+1}\sin 2^m\theta}$, where $2^m\theta \neq k\pi$, $n, m, k \in I$.

Solve the following:

**41.** $\sin\frac{9\pi}{14}.\sin\frac{11\pi}{14}\sin\frac{13\pi}{14} =$

(a) $\frac{1}{64}$ (b) $-\frac{1}{64}$ (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

**42.** $\cos 2^3\frac{\pi}{10}\cos 2^4\frac{\pi}{10}\cos 2^5\frac{\pi}{10}\ldots\cos 2^{10}\frac{\pi}{10} =$

(a) $\frac{1}{128}$ (b) $\frac{1}{256}$

(c) $\frac{1}{512}\sin\frac{\pi}{10}$ (d) $\frac{\sqrt{5}-1}{512}\sin\frac{3\pi}{10}$

**43.** $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\ldots\cos\frac{11\pi}{11} =$

(a) $-\frac{1}{32}$ (b) $\frac{1}{512}$ (c) $\frac{1}{1024}$ (d) $-\frac{1}{2048}$

**Paragraph for Question No. 44 & 45**

In triangle $ABC$, $BC = a$, $CA = b$, $AB = c$. $R$ is the circumradius and $r$ is inradius and $s$ is the semi perimeter and it is given that

$$\left(\cot\frac{A}{2}\right)^2 + \left(2\cot\frac{B}{2}\right)^2 + \left(3\cot\frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2.$$

**44.** The ratio of the sides of triangle $ABC$, $a : b : c$ is

(a) 1 : 1 : 1 (b) 13 : 40 : 45

(c) 1 : 2 : 3 (d) 7 : 15 : 45

**45.** The greatest angle of the triangle is

(a) $\pi - \cos^{-1}\left(\frac{16}{65}\right)$ (b) $\sin^{-1}\left(\frac{16}{65}\right)$

(c) $\cos^{-1}\left(\frac{63}{65}\right)$ (d) None of these

## SECTION-IV

### Matrix-Match Type

**46.** Match the following Column- I (Equations) with Column – II (No. of solutions) :

| Column I | | Column II | |
|---|---|---|---|
| (A) | $\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}x + 4$ | (p) | 4 |
| (B) | $\sin^4 x = 1 + \tan^8 x$ | (q) | 1 |
| (C) | $\cos 2x = \lvert\sin x\rvert, x \in \left(-\frac{\pi}{2}, \pi\right)$ | (r) | 3 |
| (D) | If $m$ and $n$ ($>m$) are positive integers, the no. of solutions of the equation $n\lvert\sin x\rvert = m\lvert\cos x\rvert$ in $[0, 2\pi]$ is | (s) | 0 |

**47.** Match the statements/expressions in Column I with Column II.

| Column I | | Column II | |
|---|---|---|---|
| (A) | The number of solutions of the equation $\tan x + \sec x = 2\cos x$ in the interval $[0, 2\pi]$ is | (p) | 4 |
| (B) | The number of real solutions of the equation $x^{2\log_x(x+3)} = 16$ is | (q) | $\frac{\pi}{2}$ |
| (C) | In any triangle, the area $A \le \frac{b^2 + c^2}{\lambda}$, then best possible numerical quantity $\lambda$ is | (r) | 2 |
| (D) | If the four roots of the equation $\lvert\sin\theta\rvert = k$, between $[0, 2\pi]$ are in A.P., then the common difference of A.P. is | (s) | 0 |
| | | (t) | $\pi/3$ |

## SECTION-V

### Integer Answer Type

**48.** Let $x$ be in radians with $0 < x < \frac{\pi}{2}$. If $\sin(2\sin x) = \cos(2\cos x)$; then $\tan x + \cot x$ can be written as $\frac{a}{\pi^c - b}$ where $a, b, c \in N$. Then the value of $\left(\frac{a+b+c}{25}\right)$ is

**49.** The positive integer value of $n > 3$ satisfying the equation $\frac{1}{\sin\frac{\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{\sin\frac{3\pi}{n}}$ is

**50.** $2\cot^2 x - 5\operatorname{cosec} x$ is equal to 1 for exactly 7 distinct values of $x \in [0, n\pi]$, then the greatest value of $n$ is .....

**51.** The number of solutions of the equation

$\sin^5 x - \cos^5 x = \frac{1}{\cos x} - \frac{1}{\sin x} (\sin x \neq \cos x)$ is

**52.** The number of ordered pairs $(x, y)$ where $x, y \in [0, 10]$ satisfying $\left(\sqrt{\sin^2 x - \sin x + \frac{1}{2}}\right).2^{\sec^2 y} \leq 1$ is $2K$ then $K =$

**53.** If $a, b, c$ are sides of a triangle satisfying $a^2 + b^2 + c^2 = 6$, then the A.M. of all the integral values which lie in the interval of $ab + bc + ca$ is

**54.** In $\Delta ABC$, if $R(a + b) = c\sqrt{ab}$ and $a = 2 + \sqrt{2}$, then inradius $r$ is

**55.** In $\Delta ABC$ if $\cos A + 2\cos B + \cos C = 2$, then the value of $\frac{2s}{b}$ (where '$s$' is the semi perimeter) is

**56.** Points $D, E$ are taken on the side $BC$ of an acute angled $\Delta ABC$, such that $BD = DE = EC$. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$, then the value of

$\frac{\sin(x+y)\cdot\sin(y+z)}{\sin x \cdot \sin z}$ is

## SOLUTIONS

**1. (b) :** $\frac{1+\tan\frac{y}{2}}{1-\tan\frac{y}{2}} = \left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)^3$

Square both sides, we get

$\frac{1+\sin y}{1-\sin y} = \frac{(1+\sin x)^3}{(1-\sin x)^3}$

Using componendo and dividendo

$\frac{2\sin y}{2} = \frac{(3+\sin^2 x)}{1+3\sin^2 x}\sin x$

**2. (a) :** $16(\sin^5 x + \cos^5 x) - 11(\sin x + \cos x) = 0$

$\Rightarrow (\sin x + \cos x)\{16(\sin^4 x - \sin^3 x\cos x + \sin^2 x\cos^2 x - \sin x\cos^3 x + \cos^4 x) - 11\} = 0$

$\Rightarrow (\sin x + \cos x)\{16(1 - \sin^2 x\cos^2 x - \sin x\cos x) - 11\} = 0$

$\Rightarrow (\sin x + \cos x)(4\sin x\cos x - 1)(4\sin x\cos x + 5) = 0$

As $4\sin x\cos x + 5 \neq 0$, we have

$\sin x + \cos x = 0$, $4\sin x\cos x - 1 = 0$

The required values are $\pi/12$, $5\pi/12$, $9\pi/12$, $13\pi/12$, $17\pi/12$, $21\pi/12$. There are 6 solutions in $[0, 2\pi]$.

**3. (a) :** $2\sin^2 x + \frac{2\sin x\cos x}{2} = n$

$\sin 2x - 2\cos 2x = 2n - 2$

$-\sqrt{5} \leq 2n - 2 \leq \sqrt{5} \Rightarrow 1 - \frac{\sqrt{5}}{2} \leq n \leq 1 + \frac{\sqrt{5}}{2}$

**4. (c) :** $\sin^2\left(\frac{x}{2}\right) = \frac{x}{(2n+1)\pi}$, the graph of $\sin^2\left(\frac{x}{2}\right)$ will be above the $x$–axis and will be meeting the $x$-axis at 0, $2\pi$, $4\pi$, . . . etc. It will attain maximum values at odd multiples of $\pi$ *i.e.* $\pi$, $3\pi$, ..., $(2n + 1)\pi$. The last point after which graph of $y = \frac{x}{(2n+1)\pi}$ will stop cutting will be $(2n + 1)\pi$.

Total intersection = $2(n + 1)$

**5. (a) :** $\sin A = \sin B \Rightarrow \sin A - \sin B = 0$

$\Rightarrow 2\sin\frac{A-B}{2}\cos\frac{A+B}{2} = 0$ ...(1)

and $\cos A = \cos B \Rightarrow \cos B - \cos A = 0$

$\Rightarrow 2\sin\frac{A+B}{2}\sin\frac{A-B}{2} = 0$ ...(2)

Equations (1) and (2) are simultaneously true if $\sin(1/2)(A - B) = 0$, while the other factors $\sin(1/2)(A + B)$ and $\cos(1/2)(A + B)$ both cannot be zero simultaneously.

**6. (d) :** $\log_{0.5}\sin x = 1 - \log_{0.5}\cos x$, $x \in [-2\pi, 2\pi]$

$\sin x > 0$ and $\cos x > 0$

$\sin x\cos x = \frac{1}{2}$

$\Rightarrow \sin 2x = 1 \Rightarrow 2x \in [-4\pi, 4\pi]$

$\Rightarrow$ 4 solutions.

**7. (a) :** $\sin^2 x - 2\sin x + 5 = (\sin x - 1)^2 + 4 \geq 4$

$\therefore 2^{\sqrt{\sin^2 x - 2\sin x + 5}} \geq 2^2 = 4$

and $\sin^2 y \le 1 \Rightarrow \dfrac{1}{4^{\sin^2 y}} \ge \dfrac{1}{4}$

$\therefore$ L.H.S. $\ge 1$ and according to question L.H.S. $\le 1$, therefore, L.H.S. = 1 for which

$\sin^2 x - 2\sin x + 5 = 4$

and $\sin^2 y = 1$ or $\cos y = 0$

$\Rightarrow (\sin x - 1)^2 = 0$ $\Rightarrow y = (2m+1)\dfrac{\pi}{2}$

$\Rightarrow \sin x = 1 \Rightarrow x = (4n+1)\dfrac{\pi}{2}$

**8. (d) :** $1 + \cos px + 1 + \cos qx = 2$

$\Rightarrow \cos\left(\dfrac{p+q}{2}\right)x \cos\left(\dfrac{p-q}{2}\right)x = 0$

$\Rightarrow x = \dfrac{(2n+1)\pi}{p+q}$ or $\dfrac{(2n+1)\pi}{p-q}$

for $n = 0, \pm 1, \pm 2, .....$

forms an A.P. with common difference $\dfrac{2\pi}{p+q}$ or $\dfrac{2\pi}{p-q}$.

**9. (c) :** $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$

$= \dfrac{1}{2}\left[1 + \cos 20° - (\cos 60° + \cos 40°) + (1 + \cos 100°)\right]$

$= \dfrac{1}{2}\left[1 + \cos 20° - \dfrac{1}{2} - \cos 40° + 1 - \cos 80°\right]$

$= \dfrac{1}{2}\left[\dfrac{3}{2} + \cos 20° - (2\cos 60° \cos 20°)\right] = \dfrac{3}{4}$

**10. (a) :** $1 + |\cos x| + \cos^2 x + ..... \infty$

$= \dfrac{1}{1-|\cos x|} \Rightarrow \dfrac{1}{8^{1-|\cos x|}} = 4^3$

$\Rightarrow 2^{\frac{3}{1-|\cos x|}} = 2^6 \Rightarrow \dfrac{3}{1-|\cos x|} = 6 \Rightarrow 1 - |\cos x| = \dfrac{1}{2}$

$\Rightarrow |\cos x| = \dfrac{1}{2} \Rightarrow \cos x = \pm\dfrac{1}{2}$

$\cos x = \dfrac{1}{2}$ will give least positive value of $x$

$\Rightarrow x = \dfrac{\pi}{3}$

**11. (b) :** $\dfrac{3 + \dfrac{\cos 80° \cos 20°}{\sin 80° \sin 20°}}{\dfrac{\cos 80°}{\sin 80°} + \dfrac{\cos 20°}{\sin 20°}}$

$= \dfrac{2\sin 80° \sin 20° + \cos 80° \cos 20° + \sin 80° \sin 20°}{\sin 20° \cos 80° + \cos 20° \sin 80°}$

$= \dfrac{\cos 60° - \cos 100° + \cos 60°}{\sin 100°} = \dfrac{1 - \cos 100°}{\sin 100} = \tan 50°$

**12. (d) :**

$\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$, applying $R_3 \to R_3 - R_1$

$= \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 0 & \sin\theta - 3\cos\theta & 0 \end{vmatrix}$

$= -(\sin\theta - 3\cos\theta)(3\cos\theta - \sin\theta) = (3\cos\theta - \sin\theta)^2$

Now, $-\sqrt{9+1} \le 3\cos\theta - \sin\theta \le \sqrt{9+1}$

$\Rightarrow (3\cos\theta - \sin\theta)^2 \le 10$

**13. (c) :** Given equation is

$\sec^2(a+2)x + a^2 - 1 = 0$

$\Rightarrow \tan^2(a+2)x + a^2 = 0$

$\Rightarrow \tan^2(a+2)x = 0$ and $a = 0$

$\Rightarrow \tan^2 2x = 0 \Rightarrow x = 0, \dfrac{\pi}{2}, \dfrac{-\pi}{2}$

$\therefore$ $(0, 0), (0, \pi/2), (0, -\pi/2)$ are ordered pairs satisfying the equation.

**14. (d) :** $a(b-c) + b(c-a) + c(a-b) = 0$

$\therefore$ $x = 1$ is a root of the equation

$a(b-c)x^2 + b(c-a)x + c(a-b) = 0$

Then, other root = 1 ($\because$ roots are equal)

$\therefore \alpha \times \beta = \dfrac{c(a-b)}{a(b-c)}$

$\Rightarrow ab - ac = ca - bc \quad \therefore b = \dfrac{2ac}{a+c}$

$\therefore$ $a, b, c$ are in H.P.

Then, $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P. $\Rightarrow \dfrac{s}{a}, \dfrac{s}{b}, \dfrac{s}{c}$ are in A.P.

$\Rightarrow \dfrac{s}{a} - 1, \dfrac{s}{b} - 1, \dfrac{s}{c} - 1$ are in A.P.

$\Rightarrow \dfrac{(s-a)}{a}, \dfrac{(s-b)}{b}, \dfrac{(s-c)}{c}$ are in A.P.

Multiplying each by $\dfrac{abc}{(s-a)(s-b)(s-c)}$

Then $\dfrac{bc}{(s-b)(s-c)}, \dfrac{ca}{(s-c)(s-a)}, \dfrac{ab}{(s-a)(s-b)}$ are in A.P.

$\Rightarrow \dfrac{(s-b)(s-c)}{bc}, \dfrac{(s-c)(s-a)}{ca}, \dfrac{(s-a)(s-b)}{ab}$ are in H.P.

or $\sin^2\left(\dfrac{A}{2}\right), \sin^2\left(\dfrac{B}{2}\right), \sin^2\left(\dfrac{C}{2}\right)$ are in H.P.

**15. (a) :** Given, $(\cos A + \sin A)(\cos B + \sin B) = 2$

$\cos(A - B) + \sin(A + B) = 2$

$\Rightarrow \cos(A-B)=1;\ \sin(A+B)=1$

$A=B;\ A+B=\frac{\pi}{2} \Rightarrow C=\frac{\pi}{2}$

$\therefore \frac{a+b}{c}=\sqrt{2}$

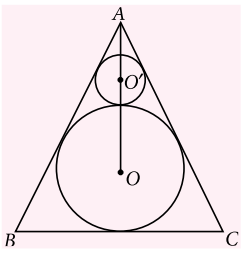
**16. (b) :** $AG=\frac{2}{3}AD=\frac{10}{3}$

$\frac{GC}{\sin\frac{\pi}{8}}=\frac{AG}{\sin\frac{\pi}{4}} \Rightarrow GC=\frac{10}{3}\times\frac{\sin\frac{\pi}{8}}{\sin\frac{\pi}{4}}$

$\therefore$ Area of $\Delta ABC = 3$ Area of $\Delta AGC$

$$3\left(\frac{1}{2}\times\frac{10}{3}\times\left(\frac{10}{3}\times\frac{\sin\frac{\pi}{8}}{\sin\frac{\pi}{4}}\right)\right)\times\sin\left(\frac{\pi}{2}+\frac{\pi}{8}\right)=\frac{25}{3}$$

**17. (d) :** Let $r_1$ be radius of required circle




$AO'=r_1\operatorname{cosec}\frac{A}{2}=\sqrt{2}r_1$

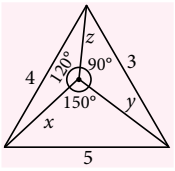
$AO=r\ \operatorname{cosec}\frac{A}{2}=\sqrt{2}r$

$OO'=\sqrt{2}\,(r-r_1)$

But $OO' = r_1 + r$

$\because\ r_1+r=\sqrt{2}(r-r_1) \Rightarrow r_1=\frac{(\sqrt{2}-1)}{(\sqrt{2}+1)}r=(3-2\sqrt{2})r$

**18. (b) :** Area of triangle $=\frac{1}{2}\times3\times4$




$=\frac{1}{2}xz\frac{\sqrt{3}}{2}+\frac{1}{2}xy\times\frac{1}{2}+\frac{1}{2}yz$

$\Rightarrow\ 24=\sqrt{3}xz+xy+2yz$

**19. (a) :** Clearly, $\angle A=\frac{\alpha}{2}, \angle B=\frac{\beta}{2}, \angle C=\frac{\gamma}{2}$

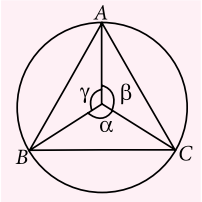
$\therefore\ \alpha+\beta+\gamma=2\pi$

$$\text{A.M.}=\frac{1}{3}\left[\cos\left(\alpha+\frac{\pi}{2}\right)+\cos\left(\beta+\frac{\pi}{2}\right)+\cos\left(\gamma+\frac{\pi}{2}\right)\right]$$

$$=-\frac{1}{3}\left[\sin\alpha+\sin\beta+\sin\gamma\right]$$

$$=-\frac{4}{3}\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$$

$$=-\frac{4}{3}\sin A\sin B\sin C$$



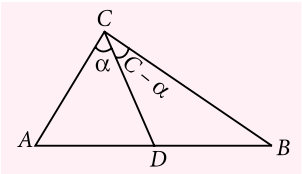

A.M. will be least if $\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$ is greatest

*i.e.* $\sin A\ \sin B\ \sin C$ is greatest, we know that in a $\Delta ABC$,

$\sin A\ \sin B\ \sin C$ is greatest if $A=B=C=\frac{\pi}{3}$

$\therefore$ Least A.M. $=-\frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)^3=-\frac{\sqrt{3}}{2}$

**20. (b) :** Let $\angle ACD = \alpha \Rightarrow \angle DCB = (C-\alpha)$




Applying the sine rule in $\Delta ACD$ and in $\Delta DCB$ respectively, we get

$$\frac{AD}{\sin\alpha}=\frac{CD}{\sin A} \text{ and } \frac{BD}{\sin(C-\alpha)}=\frac{CD}{\sin B}$$

$$\Rightarrow \frac{AD\cdot BD}{\sin\alpha\cdot\sin(C-\alpha)}=\frac{CD^2}{\sin A\cdot\sin B}$$

Now, $\frac{1}{2}\left[\cos(2\alpha-C)-\cos C\right]$

$$=\frac{1}{2}\left[\cos(2\alpha-C)-1+2\sin^2\frac{C}{2}\right]$$

$$=\sin^2\frac{C}{2}-\frac{1}{2}(1-\cos(2\alpha-C))$$

Since, $1-\cos(2\alpha-C)\ge 0$

$$\Rightarrow \sin A\cdot\sin B\le\sin^2\frac{C}{2}$$

and equality sign holds, if $1-\cos(2\alpha-C)=0$

$$\Rightarrow \alpha=\frac{C}{2}$$

**21. (b) :** We have, $r_a + r_b = 3R$

$$\Rightarrow \frac{\Delta}{s-a}+\frac{\Delta}{s-b}=3R=\frac{3abc}{4\Delta}\left(R=\frac{abc}{4\Delta}\right)$$

$$\Rightarrow \frac{\Delta(s-b+s-a)}{(s-a)(s-b)}=\frac{3abc}{4\Delta}$$

$$\Rightarrow \frac{c\Delta}{(s-a)(s-b)}=\frac{3abc}{4\Delta}\Rightarrow\frac{\Delta^2}{(s-a)(s-b)}=\frac{3ab}{4}$$

$\Rightarrow\ 4s\,(s-c)=3ab \Rightarrow (a+b+c)(a+b-c)=3ab$

$\Rightarrow\ (a+b)^2-c^2=3ab \Rightarrow a^2+b^2-c^2=ab$

$\Rightarrow\ c^2=a^2+b^2-ab$

$\Rightarrow\ a^2+b^2-2ab\cos C=a^2+b^2-ab$

(As $c^2=a^2+b^2-2ab\cos C$)

$$\Rightarrow \cos C=\frac{1}{2}\Rightarrow\angle C=60^\circ$$

Clearly, from $r_b + r_c = 2R$

$\Rightarrow \quad \frac{\Delta}{s-b}+\frac{\Delta}{s-c}=2R$

$\Rightarrow \quad \frac{\Delta(2s-b-c)}{(s-b)(s-c)}=\frac{2abc}{4\Delta} \Rightarrow \frac{2\Delta^2}{(s-b)(s-c)}=bc$

$\Rightarrow \quad 2s(s-a) = bc \Rightarrow (b+c+a)(b+c-a) = 2bc$

$\Rightarrow \quad (b+c)^2 - a^2 = 2bc$

**22. (a) :** Here $a(\cos B \cos C + \sin B \sin C) + .....$

Using $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$

$a\left(\cos B\cos C+\frac{bc}{4R^2}\right)+.....$

$=\frac{3abc}{4R^2} + a\cos B\cos C + b\cos C\cos A$

$+ c\cos A\cos B = \frac{3abc}{4R^2} + c\cos C + c\cos A\cos B$

$=\frac{3abc}{4R^2} + c\,[\cos A\cos B - \cos(A+B)]$

$=\frac{3abc}{4R^2} + c\sin A\sin B = \frac{3abc}{4R^2}+\frac{abc}{4R^2}=\frac{abc}{R^2}$

**23. (b) :** L.H.S. $=\frac{1}{2}(b+b\cos A+a+a\cos B)$

$\Rightarrow \quad \frac{1}{2}(a+b+c)=\frac{3}{2}c \Rightarrow 2c=a+b$

$\frac{a+c}{2c-a}+\frac{b+c}{2c-b}=\frac{a+c}{b}+\frac{b+c}{a}=\frac{a}{b}+\frac{b}{a}+\frac{c}{a}+\frac{c}{b}$

$\geq 4\left(\frac{c^2}{ab}\right)^{1/4}$

$\geq 4$

**24. (c) :** $\because \quad \angle A = 20^\circ \quad \therefore \angle B = \angle C = 80^\circ$

Then, $b = c$

$\therefore \quad \frac{a}{\sin 20^o}=\frac{b}{\sin 80^o}=\frac{c}{\sin 80^o}$ or $\frac{a}{\sin 20^o}=\frac{b}{\cos 10^o}$

$\Rightarrow \quad a = 2b\sin 10^\circ$

$\therefore \quad a^3 + b^3 = 8b^3\sin^3 10^\circ + b^3$

$= b^3\{2(4\sin^3 10^\circ) + 1\} = b^3\{6\sin 10^\circ\} = 3ac^2$

**25. (d) :** $r - r_2 = r_3 - r_1$

$\frac{\Delta}{s}-\frac{\Delta}{s-b}=\frac{\Delta}{s-c}-\frac{\Delta}{s-a} \Rightarrow \frac{-b}{s(s-b)}=\frac{-a+c}{(s-a)(s-c)}$

$\frac{(s-a)(s-c)}{s(s-b)}=\frac{a-c}{b} \Rightarrow \tan^2(B/2)=\frac{a-c}{b}$

But $\frac{B}{2}\in\left(\frac{\pi}{6},\frac{\pi}{4}\right) \Rightarrow \tan^2\frac{B}{2}\in\left(\frac{1}{3},1\right)$

$\Rightarrow \quad \frac{1}{3}<\frac{a-c}{b}<1$

$\Rightarrow \quad b < 3a - 3c < 3b$

$\Rightarrow \quad b + 3c < 3a < 3b + 3c$

**26. (c) :** $\cot\frac{A}{2}=\frac{b+c}{a} \Rightarrow \frac{\cos A/2}{\sin A/2}=\frac{\sin B+\sin C}{\sin A}$

$\Rightarrow \quad \frac{\cos A/2}{\sin A/2}=\frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}$

$\Rightarrow \quad \cos\frac{A}{2}=\cos\left(\frac{B-C}{2}\right) \Rightarrow \frac{A}{2}=\frac{B-C}{2}$

$\Rightarrow \quad A = B - C \Rightarrow A + C = B$

But $A + B + C = \pi$. Therefore, $B=\frac{\pi}{2}$.

**27. (c) : 1st solution:**

$2s(2s - 2a) = \lambda bc$

*i.e.*, $4\frac{s(s-a)}{bc}=\lambda$ *i.e.*, $\cos^2\frac{A}{2}=\frac{\lambda}{4}$

$\therefore \quad 0<\frac{\lambda}{4}<1$ *i.e.* $0<\lambda<4$

**2nd solution:**

$(b + c)^2 - a^2 = \lambda bc$

$b^2 + c^2 - a^2 = (\lambda - 2)bc$

$\frac{b^2+c^2-a^2}{2bc}=\frac{\lambda-2}{2}$ *i.e.* $\cos A=\frac{\lambda-2}{2}$

$\therefore \quad -1<\frac{\lambda-2}{2}<1$ *i.e.* $-2<\lambda-2<2$

*i.e.*, $0 < \lambda < 4$

**28. (a, b, c, d) :** $(1 + k)\tan^2 x - 4\tan x - 1 + k = 0$ ...(1)

Since, roots are real, we have

$(-4)^2 - 4(1 + k)(-1 + k) \geq 0$

$\Rightarrow \quad 16 - 4(k^2 - 1) \geq 0 \Rightarrow k^2 \leq 5$

We have, $\tan x_1 + \tan x_2 = \frac{4}{1+k}$

And $\tan x_1 \cdot \tan x_2 = \frac{-1+k}{1+k}$

$\therefore \quad \tan(x_1+x_2)=\frac{\frac{4}{1+k}}{1-\left(\frac{-1+k}{1+k}\right)}=\frac{4}{2}=2$

For $k = 2$, equation (1) $\Rightarrow 3\tan^2 x - 4\tan x + 1 = 0$

$\Rightarrow \quad \tan x = 1, \frac{1}{3} \quad \therefore \quad x_1=\frac{\pi}{4}, \; x_2=\tan^{-1}\frac{1}{3}$

For $k = 1$, equation (1) $\Rightarrow 2\tan^2 x - 4\tan x = 0$

$\Rightarrow \tan x = 0,\ 2 \Rightarrow x_1 = 0,\ x_2 = \tan^{-1}2$

**29. (a, b, c) :** $|\cos 2x|^2 \geq |\sin x|^2$

$\Rightarrow \cos^2 2x \geq \dfrac{1-\cos 2x}{2} \Rightarrow (\cos 2x + 1)(2\cos 2x - 1) \geq 0$

$\Rightarrow$ either $\cos 2x = -1$ or $\cos 2x \geq \dfrac{1}{2}$

$\Rightarrow$ either $2x \in 2n\pi + \pi$ or $\left[2n\pi - \dfrac{\pi}{3}, 2n\pi + \dfrac{\pi}{3}\right], n \in I$

**30. (c, d) :** $\displaystyle\sum_{m=1}^{6} \frac{\sin\left[\left(\theta + \frac{m\pi}{4}\right) - \left(\theta + \frac{(m-1)\pi}{4}\right)\right]}{\left[\sin\left(\theta + \frac{(m-1)\pi}{4}\right).\sin\left(\theta + \frac{m\pi}{4}\right)\right]} = 4$

$\Rightarrow \cot\theta - \cot\left(\theta + \dfrac{\pi}{4}\right) + \cot\left(\theta + \dfrac{\pi}{4}\right) - \cot\left(\theta + \dfrac{2\pi}{4}\right)$

$+ \ldots + \cot\left(\theta + \dfrac{5\pi}{4}\right) - \cot\left(\theta + \dfrac{6\pi}{4}\right) = 4$

$\Rightarrow \cot\theta + \tan\theta = 4$

$\Rightarrow \theta = \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$

**31. (b, d) :** Since $\sin\theta + \sqrt{3}\cos\theta = 6x - x^2 - 11$, $0 \leq \theta \leq 4\pi$

$\Rightarrow \dfrac{1}{2}\sin\theta + \dfrac{\sqrt{3}}{2}\cos\theta = \dfrac{6x - x^2 - 11}{2}$

$\Rightarrow \sin\left(\theta + \dfrac{\pi}{3}\right) = \dfrac{6x - x^2 - 11}{2}$ ... (i)

$\because -1 \leq \sin\left(\theta + \dfrac{\pi}{3}\right) \leq 1$

$\Rightarrow -1 \leq \dfrac{6x - x^2 - 11}{2} \leq 1 \Rightarrow -2 \leq 6x - x^2 - 11 \leq 2$

**Case I :** If $6x - x^2 - 11 \leq 2$

$\Rightarrow x^2 - 6x + 13 \geq 0$

$\Rightarrow (x - 3)^2 + 4 \geq 0$, which is always true.

**Case II :** If $-2 \leq 6x - x^2 - 11$

$\Rightarrow x^2 - 6x + 9 \leq 0 \Rightarrow (x - 3)^2 \leq 0$

Which is possible only, when $x - 3 = 0$

$\therefore x = 3$

From eq. (1), $\sin\left(\theta + \dfrac{\pi}{3}\right) = -1$

$\theta + \dfrac{\pi}{3} = 2n\pi - \dfrac{\pi}{2} \Rightarrow \theta = 2n\pi - \dfrac{5\pi}{6}$

For $n = 1, 2$

$\theta = \dfrac{7\pi}{6}, \dfrac{19\pi}{6}$ $(\because 0 \leq \theta \leq 4\pi)$

$\Rightarrow (x, \theta) = \left(3, \dfrac{7\pi}{6}\right), \left(3, \dfrac{19\pi}{6}\right)$

**32. (a, b, c, d) :** $\sin x \sin y = \dfrac{\sqrt{3}}{4}$ and $\cos x \cos y = \dfrac{\sqrt{3}}{4}$

Then, $\cos x \cos y + \sin x \sin y = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \cos(x - y) = \dfrac{\sqrt{3}}{2}$

$\Rightarrow x - y = 2n\pi \pm \dfrac{\pi}{6},\ n \in I$ ...(i)

and $\cos x \cos y - \sin x \sin y = 0$

$\Rightarrow \cos(x + y) = 0$

$\Rightarrow x + y = k\pi + \dfrac{\pi}{2}, k \in I$ ...(ii)

From Eqs. (i) and (ii),

$2x = 2n\pi + k\pi \pm \dfrac{\pi}{6} + \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{2}(2n + k) \pm \dfrac{\pi}{12} + \dfrac{\pi}{4}$

$\therefore x_1 = \dfrac{\pi}{2}(2n + k) + \dfrac{\pi}{3}$ and $x_2 = \dfrac{\pi}{2}(2n + k) + \dfrac{\pi}{6}$

$\therefore y_1 = \dfrac{\pi}{6} + \dfrac{\pi}{2}(k - 2n)$ and $y_2 = \dfrac{\pi}{3} + \dfrac{\pi}{2}(k - 2n)$

**33. (a, d) :** The given equation is

$(\sqrt{3}\sin x + \cos x)^{\sqrt{\sqrt{3}\sin 2x - \cos 2x + 2}} = 4$

$\Rightarrow \left[2\sin\left(x + \dfrac{\pi}{6}\right)\right]^{\sqrt{3\sin^2 x + \cos^2 x + 2\sqrt{3}\sin x\cos x}} = 4$

$\Rightarrow \left[2\sin\left(x + \dfrac{\pi}{6}\right)\right]^{\left|2\sin\left(x + \frac{\pi}{6}\right)\right|} = 4$

Hence, $2\sin\left(x + \dfrac{\pi}{6}\right) = \pm 2 \Rightarrow \sin\left(x + \dfrac{\pi}{6}\right) = \pm 1$

$\Rightarrow x + \dfrac{\pi}{6} = 2n\pi \pm \dfrac{\pi}{2};\ x = 2n\pi \pm \dfrac{\pi}{2} - \dfrac{\pi}{6}$

Also, $x \in (-\pi, \pi)$, $x = \dfrac{\pi}{3}$ and $x = \dfrac{-2\pi}{3}$

**34. (a, b, c, d) :** $\sqrt{\dfrac{6}{5}}\left(\dfrac{1}{\sqrt{2}}\sin x + \dfrac{1}{\sqrt{3}}\cos x\right)$

$= \dfrac{\sqrt{3}}{\sqrt{5}}\sin x + \dfrac{\sqrt{2}}{\sqrt{5}}\cos x$

$= \sin x.\sin\phi + \cos x\cos\phi$, where $\sin\phi = \sqrt{\dfrac{3}{5}}, \cos\phi = \sqrt{\dfrac{2}{5}}$

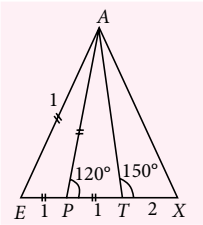
$= \cos(x - \phi) \leq 1$

$\cos^6 x + \sin^6 x = (\cos^2 x)^3 + (\sin^2 x)^3$

$$= 1 - 3\sin^2 x\cos^2 x = 1 - \frac{3}{4}(\sin 2x)^2 \le 1$$

$$\cos^2 x + \sin^4 x = 1 - \frac{(\sin 2x)^2}{4} \le 1$$

**35. (a, b, c) :**




(A) $AX^2 = AE^2 + EX^2 - 2AE \cdot EX \cos \angle AEX$

$$= 1 + 16 - 2 \cdot 1 \cdot 4 \cdot \frac{1}{2} = 13$$

$\therefore \quad AX = \sqrt{13}$

(B) Since $\Delta APT$ is isosceles $\therefore \angle ATP = \angle PAT = 30°$ then $\angle EAT = 90°$

And also $\dfrac{AT}{\sin 120^0} = \dfrac{AP}{\sin 30^0} \Rightarrow AT = \dfrac{\sqrt{3}}{2} \cdot \dfrac{1}{1/2} = \sqrt{3}$

(C) Since $EX^2 = AE^2 + AX^2 - 2AE \cdot AX \cos \angle XAE$

$16 = 1 + 13 - 2 \cdot 1\sqrt{13} \cdot \cos \angle XAE$

$$\Rightarrow \quad \cos \angle XAE = \frac{-1}{\sqrt{13}}$$

**36. (b, d) :** $\angle C = 2\angle A$ and $b = 2a \Rightarrow \sin B = 2\sin A$

$\Rightarrow \sin(A + C) = 2\sin A$

$\Rightarrow \sin 3A = 2\sin A$

$\Rightarrow 3 - 4\sin^2 A = 2$

$$\Rightarrow \sin A = \frac{1}{2} \Rightarrow A = 30°$$

$\Rightarrow C = 60°, B = 90°$

$\Rightarrow \sin^2 A + \sin^2 B + \sin^2 C = 2 \qquad ...(i)$

Now, $BC^2 + CA^2 + AB^2 = a^2 + b^2 + c^2$

$= 4R^2\sin^2 A + 4R^2\sin^2 B + 4R^2\sin^2 C$

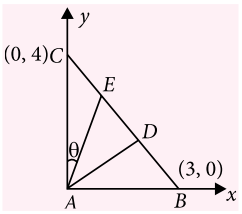
$= 4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$

$= 8R^2$ (using (i))

**37. (b, c) :** Take $A = (0, 0), B = (3, 0), C = (0, 4)$

$D, E$ are points of trisection of $BC$

$$\therefore \quad D = \left(2, \frac{4}{3}\right) \quad \therefore E = (1, 8/3)$$




$$\therefore \quad AE = \sqrt{1 + \frac{64}{9}} = \sqrt{\frac{73}{9}}$$

Slope of $\overline{AE} = \tan(90 - \theta) = \dfrac{8}{3}$

$$\Rightarrow \tan\theta = \frac{3}{8}$$

And $\cos\theta = \dfrac{8}{\sqrt{73}}$

**38. (a, b) :** $\dfrac{S + S - a + S - b + S - c}{4} \ge \sqrt[4]{S(S-a)(S-b)(S-c)}$

$$\Rightarrow \quad \frac{S}{2} \ge \Delta^{1/2} \Rightarrow \frac{S^2}{4} \ge \Delta \Rightarrow \Delta \le \frac{S^2}{4}$$

$$\frac{S - a + S - b + S - c}{3} \ge \sqrt[3]{(S-a)(S-b)(S-c)}$$

$$\frac{S^3}{27} \ge \frac{(S-a)(S-b)(S-c)S}{S} \quad \Rightarrow \quad \Delta^2 \le \frac{S^4}{27} \quad \Rightarrow \quad \Delta \le \frac{S^2}{3\sqrt{3}}$$

**(39 - 40) :**

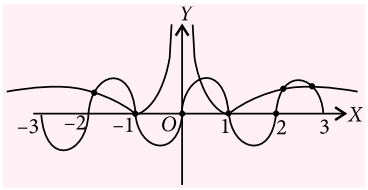
**39. (a) :** As $0 \le |\cos x| \le 1$, we have $0 \le 2[x] \le 1$

$\Rightarrow \quad [x] = 0 \quad \Rightarrow \quad 0 \le x < 1$

$$|\cos x| = 0 \quad \Rightarrow \quad x = 2n\pi + \frac{\pi}{2} \text{ but then}$$

No solution in [0, 1).

**40. (b) :** No of solutions = 6




**(41 - 43) :**

**41. (c) :** $\sin\dfrac{9\pi}{14}\sin\dfrac{11\pi}{14}\sin\dfrac{13\pi}{14} = \sin\dfrac{5\pi}{14}\sin\dfrac{3\pi}{14}.\sin\dfrac{\pi}{14}$

$$= \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = -\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$$

$$= -\frac{\sin\dfrac{8\pi}{7}}{8\sin\dfrac{\pi}{7}} = \frac{1}{8}$$

**42. (b) :** $\cos 2^3\dfrac{\pi}{10}\cos 2^4\dfrac{\pi}{10}.\cos 2^5\dfrac{\pi}{10}.....\cos 2^{10}\dfrac{\pi}{10}$

$$= \frac{\sin 2^{11}\dfrac{\pi}{10}}{256\sin 2^3\dfrac{\pi}{10}} = \frac{1}{256}$$

**43. (c) :** $\cos\dfrac{\pi}{11}\cos\dfrac{2\pi}{11}\cos\dfrac{3\pi}{11}.....\cos\dfrac{11\pi}{11}$

$$= \left(\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}\right)^2$$

$$= \left(\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{4\pi}{11}\cos\frac{8\pi}{11}\cos\frac{5\pi}{11}\right)^2$$

$$= \left(\frac{\sin 16\frac{\pi}{11}}{16\sin\frac{\pi}{11}}.\cos\frac{5\pi}{11}\right)^2 = \left(\frac{-2\sin\frac{5\pi}{11}\cos\frac{5\pi}{11}}{32\sin\frac{\pi}{11}}\right)^2 = \frac{1}{1024}$$

**(44 -45) :**

**44. (b) :** $\left(\frac{s(s-a)}{\Delta}\right)^2 + 4\left(\frac{s(s-b)}{\Delta}\right)^2 + 9\left(\frac{s(s-c)}{\Delta}\right)^2 = \frac{36s^2}{49r^2}$

$\Rightarrow \frac{s^2}{\Delta^2}((s-a)^2 + 4(s-b)^2 + 9(s-c)^2) = \frac{36}{49}\frac{s^2.s^2}{\Delta^2}$

$\Rightarrow \left(\frac{s-a}{6}\right)^2 + \left(\frac{s-b}{3}\right)^2 + \left(\frac{s-c}{2}\right)^2 = \frac{s^2}{49}$

Let $\frac{s-a}{6} = l, \frac{s-b}{3} = m, \frac{s-c}{2} = n \Rightarrow s = 6l + 3m + 2n$

$\Rightarrow 49(l^2 + m^2 + n^2) = (6l + 3m + 2n)^2$

$\Rightarrow \frac{l}{6} = \frac{m}{3} = \frac{n}{2} = k$ (from cauchy's inequality)

$\therefore\ s - a = 36k,\ s - b = 9k,\ s - c = 4k$

$\Rightarrow s = 49k,\ a = 13k,\ b = 40k,\ c = 45k$

$\therefore\ a : b : c = 13 : 40 : 45$

**45. (a) :** $\cos C = \frac{13^2 + 40^2 - 45^2}{2\times 13\times 40} = \frac{-256}{2\times 13\times 40} = \frac{-16}{65}$

$C = \cos^{-1}\left(\frac{-16}{65}\right) = \pi - \cos^{-1}\left(\frac{16}{65}\right)$

**46.** A $\rightarrow$ q; B $\rightarrow$ s; C $\rightarrow$ r; D $\rightarrow$ p

(A) L.H.S. $= \sin\left(\frac{\pi x}{2\sqrt{3}}\right) \le 1$

R.H.S. $= (x - \sqrt{3})^2 + 1 \ge 1$

L.H.S. = R.H.S. = 1 at $x = \sqrt{3}$ only

(B) $\sin^4 x \le 1$, $1 + \tan^8 x \ge 1$

L.H.S. = R.H.S = 1

$\sin^4 x = 1$, $\tan^8 x = 0$

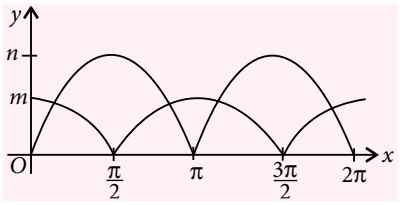
It is not possible.

(C) If $\sin x > 0$ if $\sin x < 0$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$ $\Rightarrow 2\sin^2 x - \sin x - 1 = 0$

$\Rightarrow \sin x = \frac{1}{2},\ x = \frac{\pi}{6}, \frac{5\pi}{6}$ $\Rightarrow \sin x = -\frac{1}{2}, x = -\frac{\pi}{6}$

(D) $y = n\,|\sin x|$, $y = m\,|\cos x|$




No. of points of intersection = 4

**47.** A $\rightarrow$ r; B $\rightarrow$ s; C $\rightarrow$ p; D $\rightarrow$ q

(A) $1 + \sin x = 2\cos^2 x$, $\cos x \ne 0$

$1 + \sin x = 2 - 2\sin^2 x$

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}, -1$

$\sin x \ne -1$ (as $\cos x \ne 0$) $\therefore$ it has two solutions.

(B) $x^{2\log_x(x+3)} = 16$, $x \ne 1$, $x > 0$ $\Rightarrow (x + 3)^2 = 16$

$\Rightarrow x + 3 = \pm 4$

$\Rightarrow x = 1, -7$

So, no solution.

(C) $\frac{1}{2}bc\sin A \le \frac{b^2 + c^2}{\lambda}$ $\Rightarrow \frac{1}{2}\lambda bc\sin A \le b^2 + c^2$

$bc\left[\frac{1}{2}\lambda\sin A - 2\right] \le (b - c)^2$

Since $\sin A \le 1$, the above inequality will always be satisfied if $\lambda = 4$.

(D) if $\sin\theta = \pm\frac{1}{\sqrt{2}}$

values of $\theta$ are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, which are in A.P.

**48. (2) :** $\sin(2\sin x) = \sin\left(\frac{\pi}{2} - 2\cos x\right)$

$\sin x + \cos x = \frac{\pi}{4}$

$\Rightarrow 1 + \sin 2x = \frac{\pi^2}{16}$ $\Rightarrow \sin 2x = \frac{\pi^2 - 16}{16}$

$\therefore\ \tan x + \cot x = \frac{2}{\sin 2x} = \frac{2\times 16}{\pi^2 - 16} = \frac{32}{\pi^2 - 16}$

$\therefore\ a = 32,\ b = 16,\ c = 2$

$\frac{a + b + c}{25} = 2$

**49. (7) :** $\frac{1}{\sin\theta} = \frac{1}{\sin 2\theta} + \frac{1}{\sin 3\theta}$

$\frac{1}{\sin\theta} = \frac{\sin 3\theta + \sin 2\theta}{\sin 2\theta \sin 3\theta}$

$\sin 2\theta \sin 3\theta = \sin\theta(\sin 3\theta + \sin 2\theta)$

$2\sin\theta\cos\theta\sin 3\theta = \sin\theta(\sin 3\theta + \sin 2\theta)$

$\sin 4\theta + \sin 2\theta = \sin 3\theta + \sin 2\theta$

$\Rightarrow \quad 4\theta = \pi - 3\theta \quad \Rightarrow \quad 7\theta = \pi$

$\Rightarrow \quad 7.\dfrac{\pi}{n} = \pi \quad \Rightarrow n = 7$

**50. (7) :** $2\cot^2 x - 5\operatorname{cosec} x = 1$

$\Rightarrow \quad 2\operatorname{cosec}^2 x - 5\operatorname{cosec} x - 3 = 0$

$\Rightarrow \quad (2\operatorname{cosec} x + 1)(\operatorname{cosec} x - 3) = 0$

$\operatorname{cosec} x = -\dfrac{1}{2}, \ \operatorname{cosec} x = 3$

$\operatorname{cosec} x = 3$ gives the solution in 1st and 2nd quadrant, while $\operatorname{cosec} x = -\dfrac{1}{2}$ gives no solution. So, in $[0, 2\pi]$, we get only two solutions. In $[0, 6\pi]$, we get 6 solution and between $6\pi$ and $7\pi$, we get the seventh solution. Hence, $n = 7$

**51. (0) :** Given that

$$\sin^5 x - \cos^5 x = \frac{\sin x - \cos x}{\sin x \cos x}$$

$$\sin x \cos x \left[\frac{\sin^5 x - \cos^5 x}{\sin x - \cos x}\right] = 1$$

$\sin x \cos x \{\sin^4 x + \cos^4 x + \sin^2 x \cos^2 x + \sin x \cos x(\sin^2 x + \cos^2 x)\} = 1$

$\sin x \cos x \{(\sin^2 x + \cos^2 x)^2 - \sin^2 x \cos^2 x + \sin x \cos x\} = 1$

$$\Rightarrow \quad \frac{1}{2}\sin 2x\left[1 - \frac{1}{4}\sin^2 2x + \frac{1}{2}\sin 2x\right] = 1$$

$\Rightarrow \quad \sin 2x \ (\sin^2 2x - 2\sin 2x - 4) = -8$

$\Rightarrow \quad \sin^3 2x - 2\sin^2 2x - 4\sin 2x + 8 = 0$

$\Rightarrow \quad (\sin 2x - 2)^2(\sin 2x + 2) = 0$

$\Rightarrow \quad \sin 2x = \pm 2$, which is impossible.

**52. (8) :** $$\sqrt{\sin^2 x - \sin x + \frac{1}{2}} = \sqrt{(\sin x - \frac{1}{2})^2 + \frac{1}{4}} \geq \frac{1}{2}$$

and $(\sec^2 y) \geq 1, 2^{\sec^2 y} \geq 2$

It is possible only when $\sin x = \dfrac{1}{2}, \ \sec^2 y = 1$

$\therefore \quad x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}$

$y = 0, \pi, 2\pi, 3\pi$

No. of ordered pairs = 16

**53. (5) :** $\dfrac{1}{2}\{(a-b)^2 + (b-c)^2 + (c-a)^2\} \geq 0$

$\Rightarrow \quad a^2 + b^2 + c^2 \geq ab + bc + ca \Rightarrow ab + bc + ca \leq 6$

$a^2 = b^2 + c^2 - 2bc\cos A > b^2 + c^2 - 2bc \ (\because \cos A < 1)$

$b^2 > a^2 + c^2 - 2ac$ and $c^2 > a^2 + b^2 - 2ab$

$\therefore \quad a^2 + b^2 + c^2 > 2(a^2 + b^2 + c^2 - ab - bc - ca)$

$\Rightarrow \quad ab + bc + ca > \dfrac{1}{2}(a^2 + b^2 + c^2) = 3$

Hence $ab + bc + ca \in (3, 6]$

**54. (1) :** $\dfrac{a+b}{\sqrt{ab}} = \dfrac{c}{R} \Rightarrow \dfrac{a+b}{2\sqrt{ab}} = \dfrac{c}{2R} = \sin C$

$$\cos^2 C = 1 - \frac{(a+b)^2}{4ab} = \frac{-(a-b)^2}{4ab}$$

But $\cos^2 C \geq 0 \Rightarrow a = b \Rightarrow \angle C = \dfrac{\pi}{2}$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}ab}{\frac{a+b+c}{2}} = \frac{a^2}{2a + \sqrt{2}a} = \frac{a}{2+\sqrt{2}} = \frac{2+\sqrt{2}}{2+\sqrt{2}} = 1$$

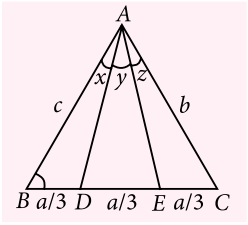
**55. (3) :** $\cos A + \cos C = 2(1 - \cos B)$

$$\Rightarrow \quad 2\cos\frac{A+C}{2}\cos\frac{A-C}{2} = 2 \times 2\sin^2\frac{B}{2}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{A-C}{2}\right)}{\sin\frac{B}{2}} = 2 \Rightarrow \frac{a+c}{b} = 2 \Rightarrow a + c = 2b$$

$\therefore \quad \dfrac{2s}{b} = \dfrac{3b}{b} = 3$

**56. (4) :**




$$\frac{a}{3\sin x} = \frac{AD}{\sin B} \qquad ...(1)$$

$$\frac{2a}{3\sin(x+y)} = \frac{AE}{\sin B} \qquad ...(2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{\sin(x+y)}{2\sin x} = \frac{AD}{AE} \qquad ...(3)$$

$$\frac{2a}{3\sin(y+z)} = \frac{AD}{\sin C} \qquad ...(4)$$

$$\frac{a}{3\sin z} = \frac{AE}{\sin C} \qquad ...(5)$$

$$\frac{(5)}{(4)} \Rightarrow \frac{\sin(y+z)}{2\sin z} = \frac{AE}{AD} \qquad ...(6)$$

$\therefore \quad (3) \times (6) \Rightarrow \dfrac{\sin(x+y)\sin(y+z)}{\sin x . \sin z} = 4$

# CONCEPT BOOSTERS

Class XII

## DIFFERENTIAL EQUATIONS

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This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

### DEFINITION

An equation involving independent variable ($x$), dependent variable ($y$) and derivative of dependent variable with respect to independent variable is called a differential equation *e.g.*

- $\frac{dy}{dx} = x \ln x$
- $dy = \cos x\, dx$
- $y = x\frac{dy}{dx} + a$
- $\left[1+\frac{dy}{dx}\right]^4 = b\left(\frac{d^2y}{dx^2}\right)^2$

### ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

The order of a differential equation is the order of the derivative of the highest order occurring in the differential equation.

The degree of a differential equation is the degree of the highest order differential coefficient appearing in it, provided it can be expressed as a polynomial equation in derivatives (*i.e.* no fractional power).

### FORMATION OF A DIFFERENTIAL EQUATION WHOSE GENERAL SOLUTION IS GIVEN:

A differential equation can be derived from its solution by the process of differentiation and other algebraical processes of elimination etc.

In order to obtain a differential equation whose solution is

$f(x, y, c_1, c_2, \ldots, c_n) = 0$

where $c_1, c_2, \ldots, c_n$ are $n$ arbitrary constants, we have to eliminate the $n$ constants for which we require $(n + 1)$ equations. The given relation along with $n$ more, obtained by successively differentiating it $n$ times, provide us the required $(n + 1)$ relations. The differential equation thus obtained is clearly of the $n^{th}$ order.

### SOLUTION OF A DIFFERENTIAL EQUATION

A solution of a differential equation is an equation which contains arbitrary constants as many as the order of the differential equation and is called the general solution. Other solutions, obtained by giving particular values to the arbitrary constants in the general solution, are called particular solutions.

Also, we know that the general integral of a function contains an arbitrary constant. Therefore, the solution of a differential equation, resulting as it does from the operations of integration, must contain arbitrary constants, equal in number to the number of times the integration is involved in obtaining the solution, and this is equal to the order of the differential equation.

Thus we see that the general solution of a differential equation of the $n^{th}$ order must contain $n$ and only $n$ independent arbitrary constants.

### SOLUTION OF A DIFFERENTIAL EQUATION BY THE METHOD OF VARIABLE SEPARATION

If the coefficient of $dx$ is only a function of $x$ and that of $dy$ is only a function of $y$ in the given

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differential equation, then the equation can be solved using variable separation method.

Thus the general form of such an equation is

$f(x)\,dx + \phi(y)\,dy = 0$ ... (i)

Integrating, we get

$\int f(x)dx + \int \phi(y)dy = C$, which is the solution of (i).

## SOLUTION OF A DIFFERENTIAL EQUATION OF THE TYPE $\frac{dy}{dx} = f(ax + by + c)$

Consider the differential equation

$$\frac{dy}{dx} = f(ax + by + c) \quad ...(i)$$

where $f(ax + by + c)$ is some function of '$ax + by + c$'.

Let $z = ax + by + c$

$$\therefore \quad \frac{dz}{dx} = a + b\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{\frac{dz}{dx} - a}{b}$$

$$\therefore \quad \text{(i)} \Rightarrow \frac{\frac{dz}{dx} - a}{b} = f(z)$$

$$\Rightarrow \frac{dz}{dx} = bf(z) + a \Rightarrow \frac{dz}{bf(z) + a} = dx \quad ...(ii)$$

In the differential equation (ii), the variables $x$ and $z$ are separated. Integrating (ii), we get

$$\int \frac{dz}{bf(z) + a} = \int dx + C$$

$$\Rightarrow \int \frac{dz}{bf(z) + a} = x + C, \text{ where } z = ax + by + c$$

This represents the general solution of the differential equation (i).

## HOMOGENEOUS EQUATION

A function $f(x, y)$ is called homogeneous function of degree $n$ if

$f(\lambda x, \lambda y) = \lambda^n f(x, y)$

For example :

- $f(x, y) = x^2y^2 - xy^3$ is a homogeneous function of degree four, since

  $f(\lambda x, \lambda y) = (\lambda^2x^2)(\lambda^2y^2) - (\lambda x)(\lambda^3y^3)$
  $= \lambda^4(x^2y^2 - xy^3) = \lambda^4 f(x, y).$

- $f(x, y) = x^2 e^{x/y} + \frac{x^3}{y} + y^2 \ln\left(\frac{y}{x}\right)$ is a homogeneous function of degree two, since

  $$f(\lambda x, \lambda y) = (\lambda^2x^2)\, e^{\lambda x/\lambda y} + \frac{\lambda^3x^3}{\lambda y} + (\lambda^2y^2)\ln\left(\frac{\lambda y}{\lambda x}\right)$$

  $$= \lambda^2\left\{x^2e^{x/y} + \frac{x^3}{y} + y^2\ln\left(\frac{y}{x}\right)\right\} = \lambda^2 f(x, y)$$

A differential equation of the form $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ is a homogenous polynomial of degree zero is called a homogenous differential equation. Such equations are solved by substituting $v = y/x$ (or $x/y$) and then separating the variables.

## DIFFERENTIAL EQUATION REDUCIBLE TO HOMOGENEOUS FORMS

Equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}\,(aB \neq Ab)$ can be reduced to a homogeneous form by changing the variables $x, y$ to $X, Y$ by substituting $x = X + h$, $y = Y + k$, where $h, k$ are constants to be chosen so as to make the given equation homogeneous, we have

$$\frac{dy}{dx} = \frac{d(Y + k)}{dx} = \frac{dY}{dx} = \frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{dY}{dX}$$

So equation becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{AX + BY + (Ah + Bk + C)}$$

Let $h$ and $k$ be chosen so as to satisfy the equations

$ah + bk + c = 0$ and

$Ah + Bk + C = 0$

This gives $h = \frac{bC - Bc}{aB - Ab}$, $k = \frac{Ac - aC}{aB - Ab}$, which are meaningful except when $aB \neq Ab$

Thus the reduced equation is $\frac{dY}{dX} = \frac{aX + bY}{AX + BY}$, which can now be solved by means of the substitution $Y = VX$.

## LINEAR DIFFERENTIAL EQUATION

Form : $\frac{dy}{dx} + Py = Q$, where $P, Q$ are functions of $x$ alone.

Integrating factor = $e^{\int P dx}$

Multiplying the form by $e^{\int P dx}$ on both sides, we get

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Q\cdot e^{\int P dx}$$

or $$\frac{dy}{dx}\cdot e^{\int P dx} + y\cdot\frac{d}{dx}e^{\int P dx} = Qe^{\int P dx}$$

$$\left[\text{since } \frac{d}{dx}(e^{\int P dx}) = P\cdot e^{\int P dx}\right]$$

or $\frac{d}{dx}(y \cdot e^{\int P dx}) = Q \cdot e^{\int P dx}$

or $\int \frac{d}{dx} \cdot (y \cdot e^{\int P dx}) dx = \int Q \cdot e^{\int P dx} dx$

or $y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} + C$

which is the required solution of the given differential equation.

## DIFFERENTIAL EQUATION REDUCIBLE TO THE LINEAR FORM

$$f'(y)\frac{dy}{dx} + f(y)P(x) = Q(x) \qquad \text{... (i)}$$

The transformation

$f(y) = u \implies f'(y)dy = du$

The equation (i) reduces to

$$\frac{du}{dx} + uP(x) = Q(x)$$

which is of the linear differential equation form.

## GENERAL FORM OF VARIABLE SEPARATION

If we can write the differential equation in the form

$f(f_1(x, y))\, d(f_1(x, y)) + \phi(f_2(x, y))\, d(f_2(x, y)) + \ldots = 0$,

then each term can be easily integrated separately.

For this the following results must be memorized.

- $d(x + y) = dx + dy$
- $d(xy) = ydx + x\,dy$
- $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$
- $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$
- $d(\log xy) = \frac{ydx + xdy}{xy}$
- $d\left(\log \frac{y}{x}\right) = \frac{(xdy - ydx)}{xy}$
- $d\left(\tan^{-1} \frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$
- $d\left(\sqrt{x^2 + y^2}\right) = \frac{xdx + ydy}{\sqrt{x^2 + y^2}}$
- $\frac{xdy - ydx}{x^2 + y^2} = \frac{\frac{xdy - ydx}{x^2}}{1 + \frac{y^2}{x^2}} = \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$
- $\frac{dx + dy}{x + y} = d[\ln(x + y)]$
- $d\left(\frac{1}{2}\ln(x^2 + y^2)\right) = \frac{x\,dx + y\,dy}{x^2 + y^2}$
- $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2 y^2}$
- $d\left(\frac{e^y}{x}\right) = \frac{x e^y dy - e^y dx}{x^2}$
- $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$
- $d(x^m y^n) = x^{m-1} y^{n-1}(my\,dx + nx\,dy)$
- $\frac{d[f(x, y)]^{1-n}}{1-n} = \frac{f'(x, y)}{(f(x, y))^n}$
- Sometimes, transformations to the polar coordinates facilitates separation of variables. It is convenient to remember the following differentials :

  If $x = r\cos\theta$, $y = r\sin\theta$,

  1. $xdx + ydy = rdr$
  2. $xdy + ydx = r^2 d\theta$
  3. $dx^2 + dy^2 = dr^2 + r^2 d\theta$

## APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

### (1) Growth And Decay Problems

Let $N(t)$ denote the amount of substance (or population) that is either growing or decaying. If we assume that $dN/dt$, the time rate of change of this amount of substance, is proportional to the amount of substance present, then $dN/dt = kN$,

or $\frac{dN}{dt} - kN = 0$

Where $k$ is the constant of proportionality. We are assuming that $N(t)$ is a differentiable, hence continuous function of time.

### (2) Temperature problems

Newton's law of cooling, which is equally applicable to heating, states that the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium. Let $T$ denote the temperature of the body and let $T_m$ denote the temperature of the surrounding medium. Then the time rate of change of temperature of the body is $\frac{dT}{dt}$, and

Newton's law of cooling can be formulated as

$\frac{dT}{dt} = -k(T - T_m)$ or as $\frac{dT}{dt} + kT = kT_m$

where $k$ is a positive constant of proportionality. Once $k$ is chosen positive, the minus sign is required in Newton's law to make $\frac{dT}{dt}$ negative in a cooling process, when $T$ is greater than $T_m$, and positive in a heating process, when $T$ is less than $T_m$.

### (3) Geometrical Applications

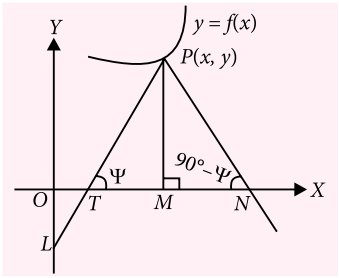
Let $P(x, y)$ be any point on the curve $y = f(x)$. Let the tangent and normal at $P(x, y)$ to the curve meets $x$–axis at $T$ and $N$ respectively.

Now draw perpendicular from $P$ on $x$–axis.

$\therefore \quad PM = y$

- If tangent at $P$ makes angle $\Psi$ with positive direction of $x$–axis, then

$$\frac{dy}{dx} = \tan\psi$$




- **Length of subtangent :** $TM$ is defined as subtangent. In $\Delta PTM$

$$TM = |y\cot\psi| = \left|\frac{y}{\tan\psi}\right| = \left|y\frac{dx}{dy}\right|$$

$\therefore$ Length of subtangent $= \left|y\frac{dx}{dy}\right|$

- **Length of subnormal :** $MN$ is defined as subnormal. In $\Delta PMN$

$$MN = |y\cot(90^0 - \psi)| = |y\tan\psi| = \left|y\frac{dy}{dx}\right|$$

$\therefore$ Length of subnormal $= \left|y\frac{dy}{dx}\right|$

- **Length of tangent :** $PT$ is defined as length of tangent. In $\Delta PMT$,

$$PT = |y\,\text{cosec}\,\psi| = \left|y\sqrt{(1+\cot^2\psi)}\right|$$

$$\therefore \text{ Length of tangent } = \left|y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\right|$$

- **Length of normal :** $PN$ is defined as length of normal. In $\Delta PMN$

$$PN = |y\,\text{cosec}(90° - \psi)| = |y\sec\psi|$$

$$= \left|y\sqrt{(1+\tan^2\psi)}\right| = \left|y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\right|$$

- **Intercepts made by the tangent on the coordinate axes :**

The equation of tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x) \qquad \ldots(i)$$

Putting $Y = 0$ in (i), we get $X = x - y\frac{dx}{dy}$

Hence, the length of intercept $OT$ that the tangent cuts off from the $x$–axis is $x - y\frac{dx}{dy}$

Again, the tangent meets $y$–axis then putting $X = 0$ in (i), we get $Y = y - x\frac{dy}{dx}$

Hence, the length of intercept $OL$ that the tangent cuts off from the $y$–axis is $y - x\frac{dy}{dx}$

## ORTHOGONAL TRAJECTORIES

The orthogonal trajectories of a family of curves form another family of curves such that each curve of one family cuts all the curves of the other family at right angles.

The differential equation of the orthogonal trajectories of the curves $f\left(x, y, \frac{dy}{dx}\right) = 0$ is the family of curves whose differential equation is $f\left(x, y, \frac{-dx}{dy}\right) = 0$

- To find the orthogonal trajectories of a family of curves whose differential equation is known, put $-\frac{dx}{dy}$ in place of $\frac{dy}{dx}$ in the equation. The resulting differential equation is the equation of the orthogonal trajectories.
- If the orthogonal trajectories form the same family of curves as the given family of curves then the given system of curves is called self-orthogonal.

## PROBLEMS

## SECTION-I

### Single Correct Answer Type

**1.** The family of curves passing through (0,0) and satisfying the differential equation $\frac{y_2}{y_1}=1$ (where $y_n = d^n y/dx^n$) is

(a) $y = k$ (b) $y = kx$

(c) $y = k(e^x + 1)$ (d) $y = k(e^x - 1)$

**2.** The degree of the differential equation satisfying the relation

$\sqrt{1+x^2}+\sqrt{1+y^2}=\lambda(x\sqrt{1+y^2}-y\sqrt{1+x^2})$ is

(a) 1 (b) 2

(c) 3 (d) none of these

**3.** $S_1$: The differential equation of parabolas having their vertices at the origin and foci on the $x$-axis is an equation whose variables are separable.

$S_2$: The differential equation of the straight lines which are at a fixed distance $p$ from the origin is an equation of degree 2.

$S_3$: The differential equation of all conics whose both axes coincide with the axes of coordinates is an equation of order 2. Which the following statements are true?

(a) All are true (b) Only $S_1$

(c) Only $S_3$ (d) Both $S_1$ and $S_2$

**4.** The order of the differential equation of all tangent lines to the parabola $y = x^2$ is

(a) 1 (b) 2 (c) 3 (d) 4

**5.** Which of the following transformation reduces the differential equation $\frac{dz}{dx}+\frac{z}{x}\ln z=\frac{z}{x^2}(\ln z)^2$ to the form $P(x)u = Q(x)$?

(a) $u = \ln z$ (b) $u = e^z$

(c) $u = (\ln z)^{-1}$ (d) $u = (\ln z)^2$

**6.** If $y_1(x)$ is a solution of the differential equation $dy/dx - f(x)y = 0$, then a solution of the differential equation $\frac{dy}{dx}+f(x)y=r(x)$ is

(a) $\frac{1}{y_1(x)}\int r(x)y_1(x)dx$ (b) $y_1(x)\int\frac{r(x)}{y_1(x)}dx$

(c) $\int r(x)y_1(x)dx$ (d) none of these

**7.** The solution of $(x + 2y^3)(dy/dx) = y$ is (where $c$ is arbitrary constant)

(a) $x = y^3 + cy$ (b) $x = y^3 - cy$

(c) $y = x^3 - cy$ (d) $y = x^3 + cy$

**8.** The solution of the equation

$$xdy=\left(y+x\frac{f(y/x)}{f'(y/x)}\right)dx \text{ is}$$

(a) $f\left(\frac{x}{y}\right)=cy$ (b) $f\left(\frac{y}{x}\right)=cx$

(c) $f\left(\frac{y}{x}\right)=cxy$ (d) $f\left(\frac{y}{x}\right)=0$

**9.** The solution of the differential equation $y_1y_3 = 3y_2^2$ is

(a) $x = A_1y^2 + A_2y + A_3$ (b) $x = A_1y + A_2$

(c) $x = A_1y^2 + A_2y$ (d) none of these

**10.** The solution of the differential equation $(x^2\sin^3y - y^2\cos x)dx + (x^3\cos y\sin^2y - 2y\sin x)dy = 0$ is

(a) $x^3\sin^3y = 3y^2\sin x + C$

(b) $x^3\sin^3y + 3y^2\sin x = C$

(c) $x^2\sin^3y + y^3\sin x = C$

(d) $2x^2\sin y + y^2\sin x = C$

**11.** If for the differential equation $y'=\frac{y}{x}+\phi\left(\frac{x}{y}\right)$, the general solution is $y=\frac{x}{\log|Cx|}$ then $\phi(x/y)$ is given by

(a) $-x^2/y^2$ (b) $y^2/x^2$

(c) $x^2/y^2$ (d) $-y^2/x^2$

**12.** A solution of $y=2x\left(\frac{dy}{dx}\right)+x^2\left(\frac{dy}{dx}\right)^4$ is

(a) $y=2c^{1/2}x^{1/4}+c$ (b) $y=2\sqrt{c}x^2+c^2$

(c) $y=2\sqrt{c}(x+1)$ (d) $y=2\sqrt{cx}+c^2$

**13.** The solution of $y_2 - 7y_1 + 12y = 0$ is

(a) $y = C_1e^{3x} + C_2e^{4x}$ (b) $y = C_1xe^{3x} + C_2e^{4x}$

(c) $y = C_1e^{3x} + C_2xe^{4x}$ (d) none of these

**14.** The curves satisfying the differential equation $(1 - x^2)y' + xy = ax$ are

(a) ellipses and hyperbolas

(b) ellipses and parabola

(c) ellipses and straight lines

(d) circles and ellipses

**15.** The solution of the differential equation $\frac{\sqrt{x}dx+\sqrt{y}dy}{\sqrt{x}dx-\sqrt{y}dy}=\sqrt{\frac{y^3}{x^3}}$ is given by

(a) $\frac{3}{2}\log\left(\frac{y}{x}\right)+\log\left|\frac{x^{3/2}+y^{3/2}}{x^{3/2}}\right|+\tan^{-1}\left(\frac{y}{x}\right)^{3/2}+c=0$

(b) $\frac{2}{3}\log\left(\frac{y}{x}\right)+\log\left|\frac{x^{3/2}+y^{3/2}}{x^{3/2}}\right|+\tan^{-1}\frac{y}{x}+c=0$

(c) $\frac{2}{3}\left(\frac{y}{x}\right)+\log\left(\frac{x+y}{x}\right)+\tan^{-1}\left(\frac{y^{3/2}}{x^{3/2}}\right)+c=0$

(d) none of these

**16.** A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is

(a) $(x - 1)^2$ (b) $(x - 1)^3$

(c) $(x + 1)^2$ (d) $(x + 1)^3$

**17.** The normal to a curve at $P(x, y)$ meets the $x$-axis at $G$. If the distance of $G$ from the origin is twice the abscissa of $P$, then the curve is

(a) a parabola (b) a circle

(c) a hyperbola (d) an ellipse

**18.** The solution of the differential equation $2x^3ydy + (1 - y^2)(x^2y^2 + y^2 - 1)dx = 0$ is [Where $c$ is a constant]

(a) $x^2y^2 = (cx + 1)(1 - y^2)$

(b) $x^2y^2 = (cx + 1)(1 + y^2)$

(c) $x^2y^2 = (cx - 1)(1 - y^2)$

(d) none of these

**19.** The solution of differential equation $(x^5 + x + 2x^2y)\,dy + (3x^4y - y)\,dx = 0$ is

(a) $x^4y + xy^2 + y = cx$ (b) $x^4y^2 + xy + y = cx$

(c) $x^4y + x^2y^2 + xy = c$ (d) $x^4y - xy^2 - y = cx$

**20.** The general solution of $x\frac{dy}{dx}+(\log x)y=x^{1-\frac{1}{2}\log x}$ is

(a) $y=x^{1-\frac{1}{2}\log x}+cx^{-\frac{1}{2}\log x}$

(b) $y.x^{\frac{1}{2}\log x}=x^{\frac{1}{2}\log x}+c$

(c) $y=e^{\frac{(\log x)^2}{2}}(x+c)$

(d) $y=e^{\frac{1}{2}(\log x)^2}(x^{1-\frac{1}{2}(\log x)}-x^{-\frac{1}{2}(\log x)})+c$

**21.** The solution of $\frac{x+y\frac{dy}{dx}}{y-x\frac{dy}{dx}}=x^2+2y^2+\frac{y^4}{x^2}$ is

(a) $\frac{y}{x}-\frac{1}{2(x^2+y^2)}=c$ (b) $\frac{y}{x}+\frac{1}{2(x^2+y^2)}=c$

(c) $\frac{x}{y}+\frac{1}{2(x^2+y^2)}=c$ (d) $\frac{x}{y}-\frac{1}{2(x^2+y^2)}=c$

**22.** The solution of $\frac{y+\sin x\cos^2(xy)}{\cos^2(xy)}dx+\left(\frac{x}{\cos^2(xy)}+\sin y\right)dy=0$ is

(a) $\tan(xy) + \cos x + \cos y = c$

(b) $\tan(xy) - \cos x - \cos y = c$

(c) $\tan(xy) + \cos x - \cos y = c$

(c) $\tan(xy) - \cos x + \cos y = c$

**23.** The equation of the curve which is passing through (1, 1) and whose differential equation is $\frac{dy}{dx}+\frac{y}{x}=y^3$ is

(a) $2x^2y^2 - xy^2 = 1$ (b) $2xy^2 + x^2y^2 = 1$

(c) $2x^2y^2 + xy^2 = 1$ (d) $2xy^2 - x^2y^2 = 1$

**24.** If the solution of differential equation $\frac{dy}{dx}=\frac{ax+3}{2y+f}$ represents a circle of non zero radius then

(a) $a = 2; 9 + 4f^2 > 4c$ (b) $a = -2; 9 + 4f^2 < 4c$

(c) $a = 2; 9 + 4f^2 < 4c$ (d) $a = -2; 9 + 4f^2 > 4c$

**25.** If the population of a country doubles in 50 years. In how many years will it become thrice the original, assume the rate of increase is proportional to the number of inhabitants.

(a) 75 (b) $50 \log_2 3$

(c) $50 \log_3 2$ (d) 100

**26.** The equation of the curve not passing through the origin and having the portion of the tangent included between the coordinate axes is bisected at the point of contact is

(a) a parabola

(b) an ellipse or a straight line

(c) a circle or an ellipse

(d) a hyperbola or a straight line

**27.** Solution of the differential equation : $\frac{dy}{dx}=\frac{3x^2y^4+2xy}{x^2-2x^3y^3}$ is

(a) $x^2y^2+\frac{x^2}{y}=c$ (b) $x^3y^2+\frac{x^2}{y}=c$

(c) $x^3y^2+\frac{y^2}{x}=c$ (d) $x^2y^3+\frac{x^2}{y}=c$

**28.** A curve is such that the mid point of the portion of the tangent intercepted between the point where the tangent is drawn and the point where the tangent meets $y$-axis, lies on the line $y = x$. If the curve passes through (1, 0), then the curve is

(a) $2y = x^2 - x$ (b) $y = x^2 - x$

(c) $y = x - x^2$ (d) $y = 2(x - x^2)$

**29.** The solution of $\frac{dy}{dx}\sqrt{1+x+y}=x+y-1$ is

(a) $2\left[\begin{array}{r}\sqrt{1+x+y}+\frac{1}{3}\log\left|\sqrt{1+x+y}-1\right| \\ -\frac{4}{3}\log\left|\sqrt{1+x+y}+2\right|\end{array}\right]=x+c$

(b) $2\left[\begin{array}{r}\sqrt{1+x+y}+\frac{1}{3}\log\left|\sqrt{1+x+y}\right| \\ -\frac{4}{3}\log\left|\sqrt{1+x+y}+2\right|\end{array}\right]=x+c$

(c) $2\left[\begin{array}{r}\sqrt{1+x+y}+\frac{1}{3}\log\left|\sqrt{1+x+y}\right| \\ -\frac{4}{3}\log\left|\sqrt{1+x+y}\right|\end{array}\right]=x+c$

(d) $\left[\begin{array}{r}\sqrt{1+x+y}+\frac{1}{3}\log\left|\sqrt{1+x+y}\right| \\ -\frac{4}{3}\log\left|\sqrt{1+x+y}\right|\end{array}\right]=x+c$

**30.** The general solution of the differential equation $\frac{dy}{dx}=y\tan x-y^2\sec x$ is

(a) $\tan x=(c+\sec x)y$ (b) $\sec y=(c+\tan y)x$
(c) $\sec x=(c+\tan x)y$ (d) none of these

**31.** The curve $y=f(x)$ is such that the area of the trapezium formed by the coordinate axes, ordinate of an arbitrary point and the tangent at this point equals half the square of its abscissa. The equation of the curve can be

(a) $y=cx^2\pm x$ (b) $y=cx^2\pm 1$
(c) $y=cx\pm x^2$ (d) $y=cx^2\pm x^2\pm 1$

**32.** If $p$ and $q$ are order and degree of differential equation $y^2\left(\frac{d^2y}{dx^2}\right)^2+3x\left(\frac{dy}{dx}\right)^{\frac{1}{3}}+x^2y^2=\sin x$, then

(a) $p>q$ (b) $\frac{p}{q}=\frac{1}{2}$
(c) $p=q$ (d) $p<q$

**33.** The solution of the differential equation $\frac{dy}{dx}-ky=0, y(0)=1$, approach zero when $x\to\infty$, if

(a) $k=0$ (b) $k>0$
(c) $k<0$ (d) none of these

**34.** The solution of $\frac{dv}{dt}+\frac{k}{m}v=-g$ is

(a) $v=ce^{-\frac{k}{m}t}-\frac{mg}{k}$ (b) $v=c-\frac{mg}{k}e^{-\frac{k}{m}t}$
(c) $ve^{-\frac{k}{m}t}=c-\frac{mg}{k}$ (d) $ve^{\frac{k}{m}t}=c-\frac{mg}{k}$

**35.** The differential equation whose solution is $(x-h)^2+(y-k)^2=a^2$ is ($a$ is a constant)

(a) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\frac{d^2y}{dx^2}$

(b) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$

(c) $\left[1+\left(\frac{dy}{dx}\right)\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$

(d) none of these

**36.** The solution of the differential equation :

$\frac{dy}{dx}=\frac{3x^2y^4+2xy}{x^2-2x^3y^3}$ is

(a) $x^2y^2+\frac{x^2}{y}=c$ (b) $x^3y^2+\frac{x^2}{y}=c$
(c) $x^3y^2+\frac{y^2}{x}=c$ (d) $x^2y^3+\frac{x^2}{y}=c$

**37.** If for a curve ratio of the distance between the normal at any of its points and the origin to the distance between the same normal and the point $(a, b)$ is equal to the constant $k$ $(k>0, k\neq 1)$, then the curve is a

(a) circle (b) parabola
(c) ellipse (d) hyperbola

**38.** Solution of $\left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx}=\left(\frac{x+y+1}{x+y+2}\right)$, given that $y=1$ when $x=1$ is

(a) $\ln\left|\frac{(x-y)^2-2}{2}\right|=2(x+y)$

(b) $\ln\left|\frac{(x+y)^2-2}{2}\right|=2(x-y)$

(c) $\ln\left|\frac{(x-y)^2+2}{2}\right|=2(x+y)$

(d) $\ln\left|\frac{(x+y)^2+2}{2}\right|=2(x-y)$

**39.** The solution of the differential equation $\left\{\frac{1}{x}-\frac{y^2}{(x-y)^2}\right\}dx+\left\{\frac{x^2}{(x-y)^2}-\frac{1}{y}\right\}dy=0$ is

(a) $\ln\left|\frac{x}{y}\right|+\frac{xy}{(x-y)}=c$ (b) $\ln|xy|+\frac{xy}{(x-y)}=c$

(c) $\frac{xy}{(x-y)}=ce^{x/y}$ (d) $\frac{xy}{(x+y)}=ce^{xy}$

## SECTION-II

### Multiple Correct Answer Type

**40.** A normal is drawn at a point $P(x, y)$ of a curve. It meet the $x$-axis at $Q$. If $PQ$ is of constant length $k$. Then

(a) The differential equation describing such curves is $y\frac{dy}{dx}=\pm\sqrt{k^2-y^2}$

(b) The curve is passing through $(0, k)$

(c) The curve is passing through $(k, 0)$

(d) The equation of the curve represents circle with centre as origin

**41.** The curve whose subtangent is '$n$' times the abscissa of the point of contact and passes through the point $(2, 3)$, then

(a) for $n = 1$, equation of the curve is $2y = 3x$

(b) for $n = 1$, equation of the curve is $2y^2 = 9x$

(c) for $n = 2$, equation of the curve is $2y = 3x$

(d) for $n = 2$, equation of the curve is $2y^2 = 9x$

**42.** If the solution of the equation $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=0$, given that for $t = 0$, $x=0$ and $\frac{dx}{dt}=12$ is in the form $x = Ae^{-3t} + Be^{-t}$, then

(a) $A + B = 0$ (b) $A + B = 12$

(c) $|AB| = 36$ (d) $|AB| = 49$

**43.** If the solution of $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=x,\ y(0)=y(1)=1$ is given by $y^2 = f(x)$ then

(a) $f(x)$ is monotonically increasing $\forall\ x \in (1, \infty)$

(b) $f(x) = 0$ has only one root

(c) $f(x)$ is neither even nor odd

(d) $f(x)$ has 3 real roots

**44.** A curve passes through $(2, 0)$ and the slope of tangent at $P(x, y)$ equals $\frac{(x+1)^2+y-3}{x+1}$ then

(a) curve is $y = x^2 - 2x$

(b) curve is $y = x^3 - 8$

(c) area bounded by the curve and $X$-axis in fourth quadrant is $\frac{2}{3}$ square units

(d) area bounded by the curve and X-axis in fourth quadrant is $\frac{4}{3}$ square units

**45.** The solution of $p^2 + (2y\cot x)p = y^2$, where $p=\frac{dy}{dx}$, is

(a) $y(1 + \cos x) = c$ (b) $y(1 - \cos x) = c$

(c) $x=2\sin^{-1}\sqrt{c/2y}$ (d) $x=2\sin^{-1}(\sqrt{2y})+c$

**46.** The function $f(\theta)=\frac{d}{d\theta}\int_0^{\theta}\frac{dx}{1-\cos\theta\cos x}$ satisfies the differential equation

(a) $\frac{df}{d\theta}+2f(\theta)\cot\theta=0$ (b) $\frac{df}{d\theta}-2f(\theta)\cot\theta=0$

(c) $f(\theta) = 1 + \cot^2\theta$ (d) $f(\theta) = 1 + \text{cosec}^2\theta$

**47.** Solution of the differential equation :

$\frac{x+y\frac{dy}{dx}}{y-x\frac{dy}{dx}}=\frac{x\sin^2(x^2+y^2)}{y^3}$ is

(a) $-\cot(x^2+y^2)=\left(\frac{x}{y}\right)^2+c$

(b) $\frac{y^2}{x^2+y^2c}=-\tan^2(x^2+y^2)$

(c) $-\cot(x^2+y^2)=\left(\frac{y}{x}\right)^2+c$

(d) $\frac{y^2+x^2c}{x^2}=-\tan^2(x^2+y^2)$

## SECTION-III

### Integer Answer Type

**48.** If the solution of the differential equation $e^{3x}(p - 1) + p^3e^{2y} = 0$ is in the form $e^y = ce^x + c^k$ then $k =$ (Where $p=\frac{dy}{dx}$ and '$c$' is arbitrary constant)

**49.** If the solution of the differential equation $y^2(y - xp) = x^4p^2$ is in the form $\frac{1}{y}=\frac{c}{x}+c^k$ then $k =$ (Where $p=\frac{dy}{dx}$ and '$c$' is arbitrary constant)

**50.** If the solution of the differential equation $y = 2px + y^2p^3$ is in the form $y^2 = 2cx + c^k$ then $k =$ (Where $p=\frac{dy}{dx}$ and '$c$' is arbitrary constant)

**51.** The solution of $x^3\frac{dy}{dx}+4x^2\tan y=e^x\sec y$ satisfying $y(1) = 0$ is $\sin y = e^x(x - 1)x^{-k}$ then $k =$

**52.** For $y > 0$ and $x \in R$, $ydx + y^2dy = xdy$ where $y = f(x)$. If $f(1) = 1$, then find $f(-3)$.

**53.** A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from normal at any point '$P$' of the curve is equal to the distance of $P$ from the $x$-axis is a circle with radius

**54.** The real value of $m$ for which the substitution $y = u^m$ will transform the differential equation $2x^4 y\frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation is $\frac{\lambda}{2}$. Then the value of $\lambda$ is

**55.** Let $y = f(x)$ be a curve passing through $(e, e^e)$ which satisfy the differential equation
$(2ny + xy\log_e x)dx - x\log_e x dy = 0$, $x > 0$, $y > 0$.

If $g(x) = \lim\limits_{n\to\infty}(0.020)f(x)$, then $\int\limits_{\frac{1}{e}}^{e} g(x)dx$ is equal to

## SOLUTIONS

**1. (d) :** $\frac{dp}{dx} = p\left(\text{where } p = \frac{dy}{dx}\right)$

$\ln p = x + c \Rightarrow p = e^{x+c}$

$\frac{dy}{dx} = ke^x \Rightarrow y = ke^x + \lambda$

Satisfying (0, 0), So $\lambda = -k$

$y = k(e^x - 1)$

**2. (a) :** On Putting $x = \tan A$, $y = \tan B$, we get
$\sec A + \sec B = \lambda(\tan A \sec B - \tan B \sec A)$
$\Rightarrow \cos A + \cos B = \lambda(\sin A - \sin B)$

$\Rightarrow \tan\left(\frac{A-B}{2}\right) = \frac{1}{\lambda}$

$\tan^{-1}x - \tan^{-1}y = 2\tan^{-1}\frac{1}{\lambda}$

On differentiating, $\frac{1}{1+x^2} - \frac{1}{1+y^2}\frac{dy}{dx} = 0$

**3. (a) :** $S_1$ - Equation of parabola is $y^2 = \pm 4ax$

$\Rightarrow 2y\frac{dy}{dx} = \pm 4a$

D.E of parabola $\Rightarrow y^2 = 2yx\frac{dy}{dx}$

$2\frac{dy}{y} = \frac{dx}{x}$

Which is variable separable.

$S_2$ - Equation of line which is at fixed distance $p$ from origin can be equation of tangent to circle $x^2 + y^2 = p^2$

Line is $y = mx + p\sqrt{1+m^2}$ $\left(m = \frac{dy}{dx}\right)$

$\left(y - x\frac{dy}{dx}\right)^2 = p\left(1 + \left(\frac{dy}{dx}\right)^2\right)$

So, degree is 2.

$S_3$ - Equation of conic whose both axes coincide with co-ordinate axes is $ax^2 + by^2 = 1$.

As there are two constants, so order of D.E. is 2.

**4. (a) :** The parametric form of the given equation is $x = t$, $y = t^2$. The equation of any tangent at $t$ is $2xt = y + t^2$, Differentiating, we get $2t = y_1\left(= \frac{dy}{dx}\right)$.

Putting this value in the above equation, we have

$$2x\frac{y_1}{2} = y + \left(\frac{y_1}{2}\right)^2 \Rightarrow 4xy_1 = 4y + y_1^2$$

The order of this equation is 1.

**5. (c) :** Given equation can be written as

$$\frac{dz}{z(\ln z)^2} + \frac{dx}{x(\ln z)} = \frac{dx}{x^2}$$

Put $u = \frac{1}{\ln z}$

$\frac{du}{dx} = -\frac{1}{(\ln z)^2}\cdot\frac{1}{z}\cdot\frac{dz}{dx}$ $\therefore$ $-\frac{du}{dx} + \frac{1}{x}.u = \frac{1}{x^2}$

$\frac{du}{dx} + \left(-\frac{1}{x}\right)u = \left(-\frac{1}{x^2}\right)$

I.F. $= e^{\int -\frac{1}{x}dx} = e^{\ln\frac{1}{x}} = \frac{1}{x}$

Hence, solution is

$u.\frac{1}{x} = \int -\frac{1}{x^2}.\frac{1}{x}dx$

$\frac{u}{x} = -\int\frac{1}{x^3}dx = \frac{1}{2x^2} + c$

**6. (a) :** $\frac{dy}{dx} - f(x).y = 0$

$\frac{dy}{y} = f(x)dx$

$\ln y = \int f(x)dx$

$y_1(x) = e^{\int f(x)dx}$

Then for given equation, I.F. $= e^{\int f(x)dx}$

Hence solution is, $y\cdot y_1(x) = \int r(x)\cdot y_1(x)dx$

or $y = \frac{1}{y_1(x)}\int r(x)\cdot y_1(x)dx$

**7. (a) :** $\frac{dx}{dy}-\frac{1}{y}x=2y^2$

This is linear equation taking $y$ as independent variable.

Here, I.F. $=e^{-\int\frac{1}{y}dy}=e^{-\log y}=\frac{1}{y}$

$\therefore$ solution is $x\frac{1}{y}=\int\frac{1}{y}2y^2dy+c$

$\Rightarrow \quad \frac{x}{y}=y^2+c \Rightarrow x=y^3+cy$

**8. (b) :** We have, $xdy=\left(y+\frac{xf(y/x)}{f'(y/x)}\right)dx$

$\Rightarrow \quad \frac{dy}{dx}=\frac{y}{x}+\frac{f(y/x)}{f'(y/x)}$, which is homogeneous

Putting $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$, we obtain

$v+x\frac{dv}{dx}=v+\frac{f(v)}{f'(v)} \Rightarrow \frac{f'(v)}{f(v)}dv=\frac{dx}{x}$

Integrating, we get $\log(f(v)) = \log x + \log c$

$\Rightarrow \quad \log f(v)=\log cx \Rightarrow f\left(\frac{y}{x}\right)=cx$

**9. (a) :** $y_1y_3=3y_2^2$

$\frac{y_3}{y_2}=3\frac{y_2}{y_1} \Rightarrow \ln y_2=3\ln y_1+\ln c$

$\Rightarrow \quad y_2=cy_1^3$

$\Rightarrow \quad \frac{y_2}{y_1^2}=cy_1 \Rightarrow -\frac{1}{y_1}=cy+c_2$

$\frac{dx}{dy}=-cy-c_2, x=-\frac{cy^2}{2}-c_2y+c_3$

$\therefore \quad x = A_1y^2 + A_2y + A_3$

**10. (a) :** $(x^2\sin^3y - y^2\cos x)dx$

$+ (x^3\cos y\sin^2y - 2y\sin x)dy = 0$

$\Rightarrow (x^3\cos y\sin^2y - 2y\sin x)\,dy - (y^2\cos x - x^2\sin^3y)dx = 0$

$\left(\frac{x^3}{3}d(\sin^3y)-\sin x\,dy^2\right)-\sin^3yd\left(\frac{x^3}{3}\right)-y^2d(\sin x)=0$

$\Rightarrow \frac{x^3}{3}d(\sin^3y)+\sin^3yd\left(\frac{x^3}{3}\right)-(\sin xdy^2+y^2d(\sin x))=0$

$\Rightarrow d\left(\frac{x^3}{3}\sin^3y\right)-d(y^2\sin x)=0$

$\Rightarrow \frac{x^3}{3}\sin^3y-y^2\sin x=c$

**11. (d) :** Putting $v = y/x$ so that $\frac{xdv}{dx}+v=\frac{dy}{dx}$, we have

$\frac{xdv}{dx}+v=v+\phi(1/v) \Rightarrow \frac{dv}{\phi(1/v)}=\frac{dx}{x}$

$\Rightarrow \log|Cx|=\int\frac{dv}{\phi(1/v)}$ ($C$ being constant of integration)

But $y=\frac{x}{\log|Cx|}$ is the general solution so

$\frac{x}{y}=\frac{1}{v}=\int\frac{dv}{\phi(1/v)}\Rightarrow \frac{1}{\phi(1/v)}=\frac{-1}{v^2}\Rightarrow \phi(x/y)=\frac{-y^2}{x^2}$

**12. (d) :** Writing $p=\frac{dy}{dx}$ and differentiating w.r.t.$x$, we have

$p=2p+2x\frac{dp}{dx}+2xp^4+4p^3x^2\frac{dp}{dx}$

$\Rightarrow \quad 0=p(1+2xp^3)+2x\frac{dp}{dx}(1+2p^3x)$

$\Rightarrow \quad p+2x\frac{dp}{dx}=0 \Rightarrow 2\frac{dp}{p}=-\frac{dx}{x}$

$\Rightarrow \quad 2\log p+\log x=\text{constant}\Rightarrow p^2x=c \text{ or } p=\sqrt{\frac{c}{x}}$

Substituting this value in the given equation, we get $y=2\sqrt{cx}+c^2$

**13. (a) :** The given equation can be written as

$\left(\frac{d}{dx}-3\right)\left(\frac{dy}{dx}-4y\right)=0$ ...(i)

If $\frac{dy}{dx}-4y=u$ then (i) reduces to $\frac{du}{dx}-3u=0$

$\Rightarrow \quad \frac{du}{u}=3dx\Rightarrow u=Ce^{3x}$

Therefore, we have $\frac{dy}{dx}-4y=Ce^{3x}$, which is a linear equation, whose I.F. is $e^{-4x}$.

So, solution is $\frac{d}{dx}(ye^{-4x})=Ce^{-x}$

$\Rightarrow \quad ye^{-4x} = -Ce^{-x} + C_2$

$\Rightarrow \quad y = C_1e^{3x} + C_2e^{4x}$

**14. (a) :** The given equation is linear in $y$ and can be written as

$\frac{dy}{dx}+\frac{x}{1-x^2}y=\frac{ax}{1-x^2}$

Its integrating factor is

$e^{\int\frac{x}{1-x^2}dx}=e^{-(1/2)\log(1-x^2)}=\frac{1}{\sqrt{1-x^2}}$ if $-1 < x < 1$

and if $x^2 > 1$ then

$$\text{I.F.} = \frac{1}{\sqrt{x^2-1}}$$

Hence, solution is

$$\frac{d}{dx}\left(y\frac{1}{\sqrt{1-x^2}}\right) = \frac{ax}{(1-x^2)^{3/2}} = -\frac{1}{2}a\frac{-2x}{(1-x^2)^{3/2}}$$

$$\Rightarrow \quad y\frac{1}{\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + C \Rightarrow y = a + C\sqrt{1-x^2}$$

$\Rightarrow (y - a)^2 = C^2(1 - x^2) \Rightarrow (y - a)^2 + C^2x^2 = C^2$

Thus if $-1 < x < 1$, the given equation represents an ellipse. If $x^2 > 1$ then the solution is of the form $-(y - a)^2 + C^2x^2 = C^2$, which represents a hyperbola.

**15. (d) :** We have, $\dfrac{\sqrt{x}dx + \sqrt{y}dy}{\sqrt{x}dx - \sqrt{y}dy} = \sqrt{\dfrac{y^3}{x^3}}$

$$\Rightarrow \quad \frac{d(x^{3/2}) + d(y^{3/2})}{d(x^{3/2}) - d(y^{3/2})} = \frac{y^{3/2}}{x^{3/2}}$$

$$\Rightarrow \quad \frac{du+dv}{du-dv} = \frac{v}{u}, \text{ where } u = x^{3/2} \text{ and } v = y^{3/2}$$

$\Rightarrow udu + udv = vdu - vdv \Rightarrow udu + vdv = vdu - udv$

$$\Rightarrow \quad \frac{udu+vdv}{u^2+v^2} = \frac{vdu-udv}{u^2+v^2}$$

$$\Rightarrow \quad \frac{d(u^2+v^2)}{u^2+v^2} = -2d\tan^{-1}\left(\frac{v}{u}\right)$$

On integrating, we get

$$\log(u^2+v^2) = -2\tan^{-1}\left(\frac{v}{u}\right) + c$$

$$\Rightarrow \quad \frac{1}{2}\log(x^3+y^3) + \tan^{-1}\left(\frac{y}{x}\right)^{3/2} = \frac{c}{2}$$

**16. (b) :** Since $f''(x) = 6(x - 1)$

$\Rightarrow f'(x) = 3(x - 1)^2 + c$ (integrating) ...(i)

Also, at the point (2,1), the tangent to the graph is $y = 3x - 5$ and slope of the tangent = 3

$\Rightarrow f'(2) = 3$

$\Rightarrow 3(2 - 1)^2 + c = 3$ [From eq (i)]

$\Rightarrow 3 + c = 3 \Rightarrow c = 0$

From Eq. (i), we have

$f'(x) = 3(x - 1)^2$

$\Rightarrow f(x) = (x - 1)^3 + k$ (Integrating) ...(ii)

$\therefore \quad 1 = (2 - 1)^3 + k \Rightarrow k = 0$

Hence, the equation of the function is

$f(x) = (x - 1)^3$

**17. (c) :** The slope of the tangent $= \dfrac{dy}{dx}$

$\therefore$ The slope of the normal $= -\dfrac{dx}{dy}$

$\therefore$ The equation of the normal is $Y - y = -\dfrac{dx}{dy}(X - x)$

This meets the $x$-axis ($Y = 0$), where

$$-y = \frac{-dx}{dy}(X - x) \Rightarrow X = x + y\frac{dy}{dx}$$

$\therefore \quad G \text{ is } \left(x + y\dfrac{dy}{dx}, 0\right)$

$\therefore \quad OG = 2x \Rightarrow x + y\dfrac{dy}{dx} = 2x$

$$\Rightarrow \quad y\frac{dy}{dx} = x \Rightarrow ydy = xdx$$

[variable separable method]

$$\Rightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + \text{constant}$$ [Integrating]

$\Rightarrow x^2 - y^2 = c$, which is a hyperbola.

**18. (c) :** $\dfrac{2y}{(1-y^2)^2}\cdot\dfrac{dy}{dx} + \dfrac{y^2}{1-y^2}\dfrac{1}{x} = \dfrac{1}{x^3}$

Put $\dfrac{y^2}{1-y^2} = t \Rightarrow \dfrac{2y}{(1-y^2)^2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\Rightarrow \quad \frac{dt}{dx} + \frac{t}{x} = \frac{1}{x^3}$$

Hence, solution is

$$t \cdot x = \int \frac{1}{x^2}dx + c \quad \Rightarrow x^2y^2 = (cx - 1)(1 - y^2)$$

**19. (a) :** Dividing both sides by $x^2$

$$x^3dy + y\cdot 3x^2dx + \frac{xdy - ydx}{x^2} + 2ydy = 0$$

$$\Rightarrow \quad d(x^3y) + d\left(\frac{y}{x}\right) + d(y^2) = 0$$

$\Rightarrow x^4y + xy^2 + y = cx$

**20. (a) :** $\dfrac{dy}{dx} + \dfrac{\log x}{x}y = x^{-\frac{1}{2}\log x}$

$$\text{I.F.} = e^{\int \frac{\log x}{x}dx} = e^{\frac{(\log x)^2}{2}} = (e^{\log x})^{\frac{\log x}{2}} = x^{\frac{1}{2}\log x}$$

Hence, general solution is

$$y\cdot x^{\frac{1}{2}\log x} = \int x^{-\frac{1}{2}\log x}\cdot x^{\frac{1}{2}\log x}dx$$

$$\Rightarrow \quad y\,x^{\frac{1}{2}\log x} = x + c$$

**21. (a) :** $\dfrac{xdx+ydy}{ydx-xdy}=\dfrac{x^4+2x^2y^2+y^4}{x^2}$

$\dfrac{xdx+ydy}{(x^2+y^2)^2}=\dfrac{ydx-xdy}{x^2} \Rightarrow \dfrac{2xdx+2ydy}{2(x^2+y^2)^2}=\dfrac{ydx-xdy}{x^2}$

$\Rightarrow \dfrac{-1}{x^2+y^2}=-2\dfrac{y}{x}+c' \Rightarrow 2\dfrac{y}{x}-\dfrac{1}{x^2+y^2}=c'$

**22. (b) :** $\dfrac{y\,dx+x\,dy}{\cos^2(xy)}+\sin x dx+\sin y dy=0$

$\Rightarrow \int\dfrac{d(xy)}{\cos^2(xy)}+\int\sin x\,dx+\int\sin y\,dy=0$

$\Rightarrow \tan(xy)-\cos x-\cos y=c$

**23. (d) :** $\dfrac{dy}{dx}+\dfrac{y}{x}=y^3 \Rightarrow \dfrac{1}{y^3}\cdot\dfrac{dy}{dx}+\dfrac{1}{xy^2}=1$

Put $t=\dfrac{1}{y^2} \Rightarrow \dfrac{dt}{dx}-\dfrac{2t}{x}=-2$

Hence, solution is $t.\dfrac{1}{x^2}=-2\int\dfrac{1}{x^2}dx$

$\Rightarrow 2xy^2+cx^2y^2=1$

Since this curve passes through $(1, 1) \Rightarrow c=-1$.

**24. (d) :** $(2y+f)dy=(ax+3)dx$

On integration,

$y^2+fy+c=\dfrac{ax^2}{2}+3x \Rightarrow -\dfrac{ax^2}{2}+y^2+fy-3x+c=0$

$\Rightarrow a=-2,\ 9+4f^2-4c>0$

**25. (b) :** $p$ – Population, $y$ – population after '$t$' years

$\dfrac{dy}{dt}\propto y \Rightarrow \dfrac{dy}{dt}=ky \Rightarrow \int\dfrac{dy}{y}=\int kdt \Rightarrow \log y=kt+c$

$t=0$ and $y=p \Rightarrow \log p=0+c$

$t=50$ and $y=2p \Rightarrow \log 2p=50k+\log p$

$\Rightarrow \log\left(\dfrac{2p}{p}\right)=50k \Rightarrow k=\dfrac{\log(2)}{50}$

$t=?$ and $y=3p \Rightarrow kt=\log y-c$

$\left(\dfrac{\log 2}{50}\right)t=\log 3p-\log p;\ t=\dfrac{\log 3}{\left(\dfrac{\log 2}{50}\right)}=50\log_2 3$

**26. (d) :** Equation of line passing through $P(x_1, y_1)$

$Y-y_1=\dfrac{dy}{dx}(X-x_1),\ x\text{-int}=x_1-\dfrac{y_1}{m},\ y\text{-int}=y_1-x_1m,$

According to condition

$\left(x-\left(x-y\dfrac{dx}{dy}\right)\right)^2+y^2=\left(y-\left(y-x\dfrac{dy}{dx}\right)\right)^2+x^2$

$\Rightarrow \left(x\dfrac{dy}{dx}-y\right)\left(x\dfrac{dy}{dx}+y\right)=0$

that is $y=cx$ or $xy=c$

**27. (b) :** $x^2dy-2x^3y^3dy=3x^2y^4dx+2xydx$

$\Rightarrow x^2dy-2xydx=3x^2y^4dx+2x^3y^3dy$

$\Rightarrow \dfrac{2xydx-x^2dy}{y^2}+3x^2y^2dx+2x^3ydy=0$

$\Rightarrow d\left(\dfrac{x^2}{y}\right)+d(x^3y^2)=0$

$\Rightarrow \dfrac{x^2}{y}+x^3y^2=C$

**28. (c) :** The point on $y$-axis is $\left(0,\ y-x\dfrac{dy}{dx}\right)$

According to given condition,

$\dfrac{x}{2}=y-\dfrac{x}{2}\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx}=2\dfrac{y}{x}-1$

Putting $\dfrac{y}{x}=v$, we get

$x\dfrac{dv}{dx}=v-1 \Rightarrow \ln\left|\dfrac{y}{x}-1\right|=\ln|x|+c$

$\Rightarrow 1-\dfrac{y}{x}=x \quad (\text{as } f(1)=0)$

**29. (a) :** Putting $\sqrt{1+x+y}=v$, we have,

$x+y-1=v^2-2 \Rightarrow 1+\dfrac{dy}{dx}=2v\dfrac{dv}{dx}$

Then the given equation transforms to

$\left(2v\dfrac{dv}{dx}-1\right)v=v^2-2 \Rightarrow \dfrac{dv}{dx}=\dfrac{v^2+v-2}{2v^2}$

$\Rightarrow \int\dfrac{2v^2}{v^2+v-2}dv$

$=\int dx \Rightarrow 2\int\left[1+\dfrac{1}{3(v-1)}-\dfrac{4}{3(v+2)}\right]dv=\int dx$

$\Rightarrow 2\left[v+\dfrac{1}{3}\log|v-1|-\dfrac{4}{3}\log|v+2|\right]=x+c$

Where $v=\sqrt{1+x+y}$

**30. (c) :** We have, $\dfrac{dy}{dx}=y\tan x-y^2\sec x$

$\Rightarrow \dfrac{1}{y^2}\dfrac{dy}{dx}-\dfrac{1}{y}\tan x=-\sec x$

Putting $\dfrac{1}{y}=v \Rightarrow \dfrac{-1}{y^2}\dfrac{dy}{dx}=\dfrac{dv}{dx}$, we obtain

$\dfrac{dv}{dx}+\tan x\cdot v=\sec x$, which is linear.

$I.F. = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$

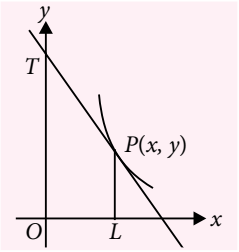
$\therefore$ The solution is

$v \sec x = \int \sec^2 x\,dx + c \Rightarrow \frac{1}{y}\sec x = \tan x + c$

$\Rightarrow$ $\sec x = y(c + \tan x)$

**31. (a) :** Let $P(x, y)$ be any point on the curve. Length of intercept on $y$-axis by any tangent at $P$ is

$$OT = y - x\frac{dy}{dx}$$




Area of trapezium $OLPTO = \frac{1}{2}(PL + OT)OL$

$$= \frac{1}{2}\left(y + y - x\frac{dy}{dx}\right)x = \frac{1}{2}\left(2y - x\frac{dy}{dx}\right)x$$

According to question

Area of trapezium $OLPTO = \frac{1}{2}x^2$

*i.e.*, $\frac{1}{2}\left(2y - x\frac{dy}{dx}\right)x = \pm\frac{1}{2}x^2$

$\Rightarrow$ $2y - x\frac{dy}{dx} = \pm x$ or $\frac{dy}{dx} - \frac{2y}{x} = \pm 1$

Which is linear differential equation.

$\text{I.F.} = e^{-2\ln x} = \frac{1}{x^2}$

The solution is $\frac{y}{x^2} = \int \pm\frac{1}{x^2}dx + c = \pm\frac{1}{x} + c$

$\therefore$ $y = \pm x + cx^2$ or $y = cx^2 \pm x$, where $c$ is an arbitrary constant.

**32. (d) :** $y^2\left(\frac{d^2y}{dx^2}\right)^2 + x^2y^2 - \sin x = -3x\left(\frac{dy}{dx}\right)^{\frac{1}{3}}$

$$\left(y^2\left(\frac{d^2y}{dx^2}\right)^2 + x^2y^2 - \sin x\right)^3 = -27x^3\left(\frac{dy}{dx}\right)$$

Here order = 2 = $p$

Degree = 6 = $q$

$\therefore$ $p < q$

**33. (c) :** $\frac{dy}{dx} - ky = 0, \frac{dy}{y} = kdx$

$\ln y = kx + c$

At $x = 0$, $y = 1$

$c = 0$

Now, $\ln y = kx$

$\Rightarrow$ $y = e^{kx}$

$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{kx} = 0$

$\therefore$ $k < 0$

**34. (a) :** $\frac{dv}{dt} + \frac{k}{m}v = -g$

Integrating factor $(\text{I.F.}) = e^{\int \frac{k}{m}dt} = e^{\frac{k}{m}t}$

$\therefore$ $ve^{\frac{k}{m}t} = -\int g\,e^{kt/m} + c$

$$ve^{\frac{k}{m}t} = \frac{-gm}{k}e^{\frac{k}{m}t} + c$$

$$v = ce^{\frac{-k}{m}t} - \frac{mg}{k}$$

**35. (b) :** $(x - h)^2 + (y - k)^2 = a^2$ ...(1)

$$2(x-h) + 2(y-k)\frac{dy}{dx} = 0 \quad \text{...(2)}$$

$$1 + \left(\frac{dy}{dx}\right)^2 + (y-k)\frac{d^2y}{dx^2} = 0 \quad \text{...(3)}$$

From (3), we have $(y - k)$, use in (2) to get $(x - h)$ and put$(x - h)$ and$(y - k)$ in(1).

**36. (b) :** The given differential equation can be written as

$(3x^2y^4 + 2xy)dx = (x^2 - 2x^3y^3)dy$

$\Rightarrow$ $3x^2y^4dx + 2x^3y^3dy + 2xy\,dx - x^2dy = 0$

$\Rightarrow$ $y^2(3x^2y^2dx + 2x^3\,ydy) + 2xy\,dx - x^2dy = 0$

$\Rightarrow$ $y^2[y^2d(x)^3 + x^3d(y)^2] + yd(x)^2 - x^2dy = 0$

$\Rightarrow$ $y^2d\,(x)^3 + x^3d\,(y^2) + \frac{yd(x)^2 - x^2dy}{y^2} = 0$

$\Rightarrow$ $d(x^3y^2) + d\left(\frac{x^2}{y}\right) = 0$

On integration, we obtain $x^3y^2 + \frac{x^2}{y} = c$, as the required solution.

**37. (a) :** Let $y = f(x)$ be the curve and let $P(x, y)$ be any point on the curve. The equation of the normal at $P(x, y)$ to the given curve is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) \quad \text{... (1)}$$

The distance of (1) from the origin is

$$d_1 = \left| \frac{y + \frac{x}{\frac{dy}{dx}}}{\sqrt{1 + \frac{1}{(dy/dx)^2}}} \right| = \left| \frac{x + y\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right|$$

$$d_2 = \left| \frac{(a-x) + (b-y)\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right|$$

Now $d_1 = kd_2$

On integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = \pm k\left[ -\frac{(a-x)^2}{2} - \frac{(b-y)^2}{2} \right] + C$$

**38. (b) :** $\because \left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx} = \left(\frac{x+y+1}{x+y+2}\right)$

Put $x + y = v$

$$\therefore \left(\frac{v-1}{v-2}\right)\left(\frac{dv}{dx} - 1\right) = \left(\frac{v+1}{v+2}\right)$$

$$\Rightarrow \frac{dv}{dx} - 1 = \frac{(v+1)(v-2)}{(v-1)(v+2)} = \frac{v^2 - v - 2}{v^2 + v - 2}$$

$$\text{or } \frac{dv}{dx} = \frac{2v^2 - 4}{v^2 + v - 2}$$

Given that $y = 1$, when $x = 1$

$$\therefore 0 + \frac{1}{2}\ln 2 = c$$

$$(y-x) + \frac{1}{2}\left|\frac{\ln(x+y)^2 - 2}{2}\right| = 0$$

$$\text{or } \ln\left|\frac{(x+y)^2 - 2}{2}\right| = 2(x-y)$$

**39. (a) :** The given equation can be written as

$$\left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{(x^2dy - y^2dx)}{(x-y)^2} = 0$$

$$\Rightarrow \left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{\left(\frac{dy}{y^2} - \frac{dx}{x^2}\right)}{\left(\frac{1}{y} - \frac{1}{x}\right)^2} = 0$$

$$\Rightarrow \left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{\frac{dy}{y^2} - \frac{dx}{x^2}}{\left(\frac{1}{x} - \frac{1}{y}\right)^2} = 0$$

Integrating, we get

$$\ln|x| - \ln|y| - \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)} = c$$

$$\text{or } \ln\left|\frac{x}{y}\right| - \frac{xy}{(y-x)} = c \text{ or } \ln\left|\frac{x}{y}\right| + \frac{xy}{x-y} = c$$

**40. (a, b, c, d) :** Equation of the normal at a point $P(x, y)$ is given by

$$Y - y = -\frac{1}{dy/dx}(X - x) \quad \ldots(1)$$

Let the point $Q$ at the $x$-axis be $(x_1, 0)$. From (1) we get

$$y\frac{dy}{dx} = x_1 - x \quad \ldots(2)$$

Now given that $PQ^2 = k^2$

We have $(x - x_1)^2 + y^2 = k^2$

$$\text{or } x - x_1 = \pm\sqrt{k^2 - y^2}$$

Hence using (2), we obtain $y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$ ...(3)

(3) is the required differential equation for such curves

Now solving (3), we get $\int \frac{-ydy}{\sqrt{k^2 - y^2}} = \int -dx$

or $x^2 + y^2 = k^2$

which is equation of circle with centre at origin passes through $(0, k)$ and $(k, 0)$

**41. (a, d) :** If $(x, y)$ is any point on the curve, the subtangent at $(x, y) = y\frac{dx}{dy}$

$$\therefore y\frac{dx}{dy} = nx \text{ (given) or } n\frac{dy}{y} = \frac{dx}{x}$$

Integrating, $n\log y = \log x + \log c$

or $\log y^n = \log cx$

or $y^n = cx$ .....(i), which is the required equation of the family of curves.

Putting $x = 2$, $y = 3$ in (i), we have $3^n = 2c$ or $c = \frac{3^n}{2}$

Putting this value of $c$ in (i), we get

$$y^n = \frac{3^n}{2}x \text{ or } 2y^n = 3^n x \quad \text{(ii)}$$

which is the particular curve passing through the point (2, 3).

Putting $n = 1$ in (ii), we have $2y = 3x$, which is a straight line.

Putting $n = 2$ in (ii), we have $2y^2 = 9x$

which is a parabola.

**42. (a, c) :** $x = Ae^{-3t} + Be^{-t}$

$$\frac{dx}{dt} = -3Ae^{-3t} - Be^{-t}$$

At $t = 0, x = 0 \Rightarrow 0 = A + B$ ...(1)

At $t = 0, \frac{dx}{dt} = 12 \Rightarrow 12 = -3A - B$ ...(2)

Solving (1) and (2), $A = -6, B = 6$

**43. (a, b, c) :** Given $\frac{d}{dx}\left(y\frac{dy}{dx}\right) = x \Rightarrow y\frac{dy}{dx} = \frac{x^2}{2} + c$

$\Rightarrow \frac{d}{dx}\left(\frac{y^2}{2}\right) = \frac{x^2}{2} + c$

$\Rightarrow y^2 = \frac{x^3}{3} + \alpha x + \beta$ given $y(0) = 1, y(1) = 1$

$\Rightarrow \beta = 1, \alpha = \frac{-1}{3}$

$\therefore y^2 = f(x) = \frac{x^3 - x + 3}{3}, f'(x) = \frac{(3x^2 - 1)}{3} > 0$ for $x > 1$

$\therefore$ $f(x)$ is monotonically increasing $\forall\, x \in (1, \infty)$

$f\left(\frac{1}{\sqrt{3}}\right) f\left(-\frac{1}{\sqrt{3}}\right) > 0$

$f(x) = 0$ has only one real root.

**44. (a, d) :** $\frac{dy}{dx} = \frac{(x+1)^2 + y - 3}{x+1} = x + 1 + \frac{y-3}{x+1}$

Putting $x + 1 = X$ and $y - 3 = Y$

$\frac{dY}{dX} = X + \frac{Y}{X} \Rightarrow \frac{dY}{dX} - \frac{Y}{X} = X$

Solution is $\frac{Y}{X} = X + c$

$\Rightarrow y - 3 = (x + 1)^2 + c(x + 1)$

It passes through $(2, 0) \Rightarrow c = -4$

Equation of curve is $y = x^2 - 2x$

Area bounded $= \int_0^2 (2x - x^2)dx = \frac{4}{3}$ sq. units

**45. (a, b, c) :** $p^2 + 2py\cot x - y^2 = 0$

$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$

$\Rightarrow \frac{dy}{dx} = (-\cot x \pm \operatorname{cosec} x)y \Rightarrow \frac{dy}{dx} + y\frac{(\cos x \pm 1)}{\sin x} = 0$

On integration

$\log y + 2\log\sin\frac{x}{2} = \log c'$

$\Rightarrow y(1 - \cos x) = 2c' = c$ and $y(1 + \cos x) = c$

$\Rightarrow y\left(2\sin^2\frac{x}{2}\right) = c \Rightarrow \sin^2\frac{x}{2} = \frac{c}{2y}$

$\Rightarrow x = 2\sin^{-1}\sqrt{\frac{c}{2y}}$

**46. (a, c) :** $f(\theta) = \frac{d}{d\theta}\int_0^\theta \frac{dx}{1 - \cos x\cos\theta} = \frac{1}{1 - \cos^2\theta} = \operatorname{cosec}^2\theta$

$\Rightarrow \frac{df}{d\theta} = -2\operatorname{cosec}^2\theta\cot\theta$

$\frac{df}{d\theta} + 2f(\theta)\cot\theta = 0$ and $f(\theta) = \operatorname{cosec}^2\theta$

$= 1 + \cot^2\theta$

**47. (a, b) :** The given differential equation can be written as

$\frac{xdx + ydy}{ydx - xdy} = \frac{x\sin^2(x^2 + y^2)}{y^3}$

$\Rightarrow \frac{2xdx + 2ydy}{ydx - xdy} = \frac{2x\sin^2(x^2 + y^2)}{y^3}$

$\Rightarrow \frac{d(x^2 + y^2)}{\sin^2(x^2 + y^2)} = \frac{2x}{y^3}(ydx - xdy)$

$\Rightarrow \operatorname{cosec}^2(x^2 + y^2)\, d(x^2 + y^2) = 2\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right)$

On integrating, we get

$\int \operatorname{cosec}^2(x^2 + y^2)\, d(x^2 + y^2) = 2\int\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right)$

$\Rightarrow -\cot(x^2 + y^2) = \left(\frac{x}{y}\right)^2 + c,$

which is the required solution.

**48. (3) :** Put $e^x = X$ and $e^y = Y$

So that $e^x dx = dX$ and $e^y dy = dY$

$\Rightarrow \frac{e^y}{e^x}\cdot\frac{dy}{dx} = \frac{dY}{dX}$ or $\frac{Y}{X}p = P$ (say) or $p = \frac{X}{Y}\cdot P$

The given equation becomes

$X^3\left(\frac{X}{Y}P - 1\right) + \frac{X^3}{Y^3}.Y^2P^3 = 0$

or $XP - Y + P^3 = 0$ or $Y = PX + P^3$

Which is the Clairaut's form.

$\therefore$ The solution is $Y = cX + c^3$ or $e^y = ce^x + c^3$

**49. (2) :** Put $x = \frac{1}{X}$ and $y = \frac{1}{Y}$

$dx = -\frac{1}{X^2}dX$ and $dy = -\frac{1}{Y^2}dY$

$\Rightarrow p = \frac{dy}{dx} = \frac{X^2}{Y^2}\frac{dY}{dX} = \frac{X^2}{Y^2}P$

The given equation becomes

$\frac{1}{Y^2}\left(\frac{1}{Y} - \frac{1}{X}\cdot\frac{X^2}{Y^2}P\right) = \frac{1}{X^4}\cdot\frac{X^4}{Y^4}P^2$

$\Rightarrow$ $Y - XP = P^2$ or $Y = PX + P^2$
which is the Clairaut's form
$\therefore$ The solution is $Y = cX + c^2$
or $\frac{1}{y} = \frac{c}{x} + c^2$

**50. (3) :** The given equation is $y = 2px + y^2p^3$ ...(i)

Solving for $x$, $x = \frac{y}{2p} - \frac{1}{2}y^2p^2$

Differentiating w.r.t $y$, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\cdot\frac{dp}{dy} - yp^2 - y^2p\cdot\frac{dp}{dy}$$

or $2p = p - y\frac{dp}{dy} - 2yp^4 - 2y^2p^3\frac{dp}{dy}$

or $p(1+2yp^3) + y\frac{dp}{dy}(1+2yp^3) = 0$

or $(1+2yp^3)\left(p + y\frac{dp}{dy}\right) = 0$

Neglecting the first factor which does not involve $\frac{dp}{dy}$, we have

$p + y\frac{dp}{dy} = 0 \Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0$

Integrating, $\log p + \log y = \log c$
or $\log py = \log c \Rightarrow py = c$ ...(ii)
Eliminating $p$ between (i) and (ii)

$$y = 2x\cdot\frac{c}{y} + y^2\cdot\frac{c^3}{y^3} \quad \text{or} \quad y = \frac{2cx}{y} + \frac{c^3}{y}$$

or $y^2 = 2cx + c^3$, which is the required solution.

**51. (4) :** Put $\sin y = t \Rightarrow \cos y\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ $\frac{dt}{dx} + \frac{4}{x}t = \frac{e^x}{x^3}$

I.F. $= e^{\int(4/x)dx} = e^{4\ln x} = x^4$

$$\frac{d(t\cdot x^4)}{dx} = xe^x$$

**52. (3) :** $\frac{xdy - ydx}{y^2} = dy \Rightarrow -\int d\left(\frac{x}{y}\right) = \int dy$

$\Rightarrow -\frac{x}{y} = y + c$

At $x = 1$, $y = 1$, $c = -2$

At $x = -3$, $\frac{3}{y} = y - 2$

$\Rightarrow y^2 - 2y - 3 = 0 \quad \therefore y = 3$

**53. (1) :** Equation of normal at the point $P(x, y)$ is

$$Y - y = -\frac{dx}{dy}(X - x)\left(\text{let } m = \frac{dy}{dx}\right)$$

Let $m = \frac{dy}{dx} \Rightarrow X + mY - (x + my) = 0$ ...(i)

Distance of perpendicular from the origin to line (i) is $\frac{|x+my|}{\sqrt{1+m^2}} = |y|$

$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

This is homogeneous equation.
Let $y = zx$,

$\Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx} \Rightarrow \frac{2z}{1+z^2}dz = -\frac{dx}{x}$

Integrating $\int \frac{2z}{1+z^2}dz = -\int \frac{dx}{x}$

$\Rightarrow \log(1 + z^2) = -\log x + c \Rightarrow (x^2 + y^2) = x.e^c$
This curve passes through (1, 1)
$\Rightarrow 1 + 1 = 1\cdot e^c \Rightarrow e^c = 2$
The required equation of the curve is
$x^2 + y^2 = 2x$

**54. (3) :** $y = u^m \Rightarrow \frac{dy}{dx} = mu^{m-1}\frac{du}{dx}$

Hence $2x^4.u^m.mu^{m-1}.\frac{du}{dx} = 4x^6 - u^{4m}$

$\frac{du}{dx} = \frac{4x^6 - u^{4m}}{2mx^4u^{2m-1}} \Rightarrow 4m = 6 \Rightarrow m = \frac{3}{2}$

**55. (0) :** $(2ny + xy\log_e x)dx = x\log_e x dy$

$\Rightarrow \frac{dy}{y} = \left(\frac{2n}{x\log_e x} + 1\right)dx$

$\Rightarrow \log(y) = 2n\log|\log x| + x + c$ and $c = 0$

Now, $g(x) = (0.020)\lim_{n\to\infty} f(x) = \begin{cases} \to\infty, & \text{if } x < \frac{1}{e} \\ \to 0, & \text{if } \frac{1}{e} < x < e \\ \to\infty, & \text{if } x > e \end{cases}$

$\therefore \int_{1/e}^{e} g(x)dx = 0$

# WBJEE MOCK TEST PAPER

## UNIT-IV : CALCULUS

The entire syllabus of Mathematics of WB-JEE is being divided into six units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issue.

### CATEGORY I

**For each correct answer one mark will be awarded, whereas, for each wrong answer, 25% of total marks (1/4) will be deducted. If candidates mark more than one answer, negative marking will be done.**

**1.** A function $f$ satisfies the relation $f(n^2) = f(n) + 6$ for $n \geq 2$ and $f(2) = 8$. Then the value of $f(256)$ is

(a) 24 (b) 26 (c) 22 (d) 28

**2.** $\lim\limits_{x\to 0}\left(\frac{10\sin 9x}{9\sin 10x}\right)\left(\frac{8\sin 7x}{7\sin 8x}\right)\left(\frac{6\sin 5x}{5\sin 6x}\right)\left(\frac{4\sin 3x}{3\sin 4x}\right)\left(\frac{\sin x}{\sin 2x}\right) =$

(a) $\frac{63}{256}$ (b) $\frac{1}{6}$ (c) $\frac{6}{5}$ (d) $\frac{1}{2}$

**3.** The domain of the function $f(x) = \sqrt{7-3x} + \log_e x$ is

(a) $0 < x < \infty$ (b) $\frac{7}{3} \leq x < \infty$

(c) $0 < x \leq \frac{7}{3}$ (d) $-\infty < x < 0$

**4.** The range of the function $f(x) = \frac{1}{2-\cos 3x}$ is

(a) $(-2, \infty)$ (b) $[-2, 3]$

(c) $\left(\frac{1}{3}, 2\right)$ (d) $\left[\frac{1}{3}, 1\right]$

**5.** If $f(x + y) = f(x)f(y)$ for all $x$ and $y$ and $f(5) = 2$, $f'(0) = 3$, then $f'(5)$ is

(a) 5 (b) 6
(c) 0 (d) none of these

**6.** If $f(x) = \log_{x^2}(\log x)$, then $f'(x)$ at $x = e$ is

(a) 0 (b) 1 (c) $\frac{1}{e}$ (d) $\frac{1}{2e}$

**7.** The function $f(x) = [x]$, where $[x]$ denotes greatest integer function is continuous at

(a) $-2$ (b) 1.5 (c) 4 (d) 1

**8.** The function $f(x) = a\sin|x| + be^{|x|}$ is differentiable at $x = 0$ when

(a) $3a + b = 0$ (b) $3a - b = 0$
(c) $a + b = 0$ (d) $a - b = 0$

**9.** Let $f(x)$ be a differentiable function and $f'(4) = 5$. Then $\lim\limits_{x\to 2}\frac{f(4)-f(x^2)}{x-2}$ equals

(a) 0 (b) 5 (c) 20 (d) $-20$

**10.** The value of $\frac{d}{dx}[x(x-1)(x-2)(x-3)]$ at $x = 3$ is

(a) 0 (b) 3 (c) 6 (d) 24

**11.** $\lim\limits_{x\to 0} x\sin(e^{1/x})$

(a) is equal to 0 (b) is equal to 1
(c) is equal to $\frac{e}{2}$ (d) does not exist

**12.** If $f(x) = \sin^{-1}\left(\frac{1-\cos 2x}{2\sin x}\right)$, then $|f'(x)|$ is equal to

(a) $|\sin x|$ (b) $x$
(c) 0 (d) 1

**13.** If $|t| < 1$, $\sin x = \frac{2t}{1+t^2}$, $\tan y = \frac{2t}{1-t^2}$, then $\frac{dy}{dx} =$

(a) $\frac{1}{x}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1

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**14.** If $y=\frac{x}{x+1}+\frac{x+1}{x}$, then $\frac{d^2y}{dx^2}$ at $x=1$ is equal to

(a) $\frac{7}{4}$ (b) $\frac{7}{8}$ (c) $\frac{1}{4}$ (d) $\frac{-7}{8}$

**15.** If a circular plate is heated uniformly, its area expand $3c$ times as fast as its radius, then the value of $c$ when the radius is 6 units, is

(a) $4\pi$ (b) $2\pi$ (c) $6\pi$ (d) $3\pi$

**16.** The function $f(x) = \sin x - kx - c$, where $k$ and $c$ are constants, decreases always when

(a) $k > 1$ (b) $k \geq 1$
(c) $k < 1$ (d) $k \leq 1$

**17.** The chord joining the points (5, 5) and (11, 227) on the curve $y = 3x^2 - 11x - 15$ is parallel to tangent at a point on the curve. Then the abscissa of the point is

(a) $-4$ (b) 4
(c) $-8$ (d) 8

**18.** If $x$ is real, then the minimum value of $x^2 - 8x + 17$ is

(a) 2 (b) 4
(c) 1 (d) 3

**19.** If $y = f(x^2 + 2)$ and $f'(3) = 5$, then $\frac{dy}{dx}$ at $x = 1$ is

(a) 25 (b) 10
(c) 5 (d) 15

**20.** $\int \frac{\sin^2 x}{1+\cos x}dx =$

(a) $x - \sin x + C$ (b) $\cos x + C$
(c) $x + \sin x + C$ (d) $\sin x + C$

**21.** $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right)dx$ is

(a) $\tan\left(\frac{x}{2}\right)+C$ (b) $e^x \sin x + C$

(c) $e^x \tan\left(\frac{x}{2}\right)+C$ (d) $e^x + C$

**22.** $\int \frac{\cos 8x+1}{\tan 2x-\cot 2x}dx = a\cos 8x + C$ then $a$ is

(a) $-\frac{1}{16}$ (b) $\frac{1}{8}$

(c) $\frac{1}{16}$ (d) $-\frac{1}{8}$

**23.** The value of the integral $\int_0^{\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$ is

(a) $\pi^2$ (b) $2\pi^2$ (c) $\frac{\pi^2}{4}$ (d) $\frac{\pi^2}{2}$

**24.** If $\int_0^{\pi} xf(\sin x)dx = A\int_0^{\pi/2} f(\sin x)dx$, then $A$ is equal to

(a) 0 (b) $2\pi$ (c) $\frac{\pi}{4}$ (d) $\pi$

**25.** Area bounded by the curve $y = x^3$, the $x$-axis and the ordinates at $x = -2$ and $x = 1$, is

(a) $-9$ sq. units (b) $-\frac{15}{4}$ sq. units

(c) $\frac{15}{4}$ sq. units (d) $\frac{17}{4}$ sq. units

**26.** The area in square units bounded by the normal at (1, 2) to the parabola $y^2 = 4x$, $x$-axis and the curve is given by

(a) $\frac{10}{3}$ (b) $\frac{7}{3}$

(c) $\frac{4}{3}$ (d) none of these

**27.** Area bounded by the $y$-axis, $y = \cos x$, $y = \sin x$ when $0 \leq x \leq \frac{\pi}{2}$ is

(a) $\sqrt{2}-1$ sq. units (b) $2(\sqrt{2}-1)$ sq. units

(c) $\sqrt{2}+1$ sq. units (d) $\sqrt{2}$ sq. units

**28.** The area of the region bounded by $y^2 = 9x$ and $y = 3x$ is

(a) 2 sq. units (b) $\frac{1}{4}$ sq. unit

(c) $\frac{1}{2}$ sq. unit (d) 1 sq. unit

**29.** The solution of the differential equation $y'(y^2 - x) = y$ is

(a) $y^3 - 3xy = C$ (b) $y^3 + 3xy = C$
(c) $x^3 - 3xy = C$ (d) $y^3 - xy = C$

**30.** The order and degree of the differential equation $\left[2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^3y}{dx^3}$ are respectively

(a) 2 and 2 (b) 2 and 1
(c) 3 and 2 (d) 3 and 3

## CATEGORY II

**Every correct answer will yield 2 marks. For incorrect response, 25% of full mark (1/2) would be deducted. If candidates mark more than one answer, negative marking will be done.**

**31.** The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)}\,dx$ is equal to

(a) 1 (b) 6 (c) 2 (d) 4

**32.** The integral $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ equals

(a) $-(x^4+1)^{1/4} + C$ (b) $-\left(\frac{x^4+1}{x^4}\right)^{1/4} + C$

(c) $\left(\frac{x^4+1}{x^4}\right)^{1/4} + C$ (d) $(x^4+1)^{1/4} + C$

**33.** If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx + 2, & 3 < x \le 5 \end{cases}$ is differentiable, then the value of $k + m$ is

(a) $\frac{10}{3}$ (b) 4 (c) 2 (d) $\frac{16}{5}$

**34.** $\int_0^1 \frac{dx}{(x^2+16)(x^2+25)} =$

(a) $\frac{1}{5}\left[\frac{1}{4}\tan^{-1}\left(\frac{1}{4}\right) - \frac{1}{5}\tan^{-1}\left(\frac{1}{5}\right)\right]$

(b) $\frac{1}{9}\left[\frac{1}{4}\tan^{-1}\left(\frac{1}{4}\right) - \frac{1}{5}\tan^{-1}\left(\frac{1}{5}\right)\right]$

(c) $\frac{1}{4}\left[\frac{1}{4}\tan^{-1}\left(\frac{1}{4}\right) - \frac{1}{5}\tan^{-1}\left(\frac{1}{5}\right)\right]$

(d) $\frac{1}{9}\left[\frac{1}{5}\tan^{-1}\left(\frac{1}{4}\right) - \frac{1}{4}\tan^{-1}\left(\frac{1}{5}\right)\right]$

**35.** The value of $c$ in Lagrange's theorem for the function $f(x) = \log(\sin x)$ in the interval $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{2\pi}{3}$ (d) none of these

## CATEGORY III

**In this section more than 1 answer can be correct. Candidates will have to mark all the correct answers, for which 2 marks will be awarded. If, candidates mark one correct and one incorrect answer then no marks will be awarded. But if, candidate makes only correct, without marking any incorrect, formula below will be used to allot marks.**
**2 × (no. of correct response/total no. of correct options)**

**36.** Consider the family of all circles whose centres lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$ where $P$, $Q$ are functions of $x$, $y$ and $y'\left(\text{here } y' = \frac{dy}{dx'}\, y'' = \frac{d^2y}{dx^2}\right)$, then which of the following statement is (are) true?

(a) $P = y + x$
(b) $P = y - x$
(c) $P + Q = 1 - x + y + y' + (y')^2$
(d) $P - Q = x + y - y' - (y')^2$

**37.** Let $f(x)$ be a real valued function such that $f(0) = \frac{1}{2}$ and $f(x+y) = f(x)\,f(a-y) + f(y)\,f(a-x)\ \forall\ x, y \in R$, then for some real '$a$'

(a) $f(x)$ is periodic function
(b) $f(x)$ is a constant function
(c) $f(x) = \frac{1}{2}$ (d) $f(x) = \frac{\cos x}{2}$

**38.** If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ has a local maximum at $x = x_1$ and a local minium at $x = x_2$ such that $x_2 = x_1^2$, then $a$ is equal to

(a) 0 (b) $\frac{1}{4}$

(c) 2 (d) either (a) or (c)

**39.** The angle of intersection between the curves $y = [|\sin x| + |\cos x|]$ and $x^2 + y^2 = 10$, where $[x]$ denotes the greatest integer $\le x$, is

(a) $\tan^{-1}3$ (b) $\tan^{-1}(-3)$

(c) $\tan^{-1}\sqrt{3}$ (d) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

**40.** If $g(x) = \int_0^x \cos 4t\,dt$, then $g(x+\pi)$ equals

(a) $g(x) - g(\pi)$ (b) $g(x)\cdot g(\pi)$

(c) $\frac{g(x)}{g(\pi)}$ (d) $g(x) + g(\pi)$

## SOLUTIONS

**1. (b):** $f(256) = f(16^2) = f(16) + 6$
$f(16) = f(4^2) = f(4) + 6$
$f(4) = f(2^2) = f(2) + 6$
$\because\ f(2) = 8 \quad \therefore\ f(4) = 8 + 6 = 14$
$\Rightarrow\ f(16) = 14 + 6 = 20$ and $f(256) = 20 + 6 = 26$

**2. (d) :** $\lim_{x\to 0}\left(\frac{10\sin 9x}{9\sin 10x}\right)\left(\frac{8\sin 7x}{7\sin 8x}\right)\left(\frac{6\sin 5x}{5\sin 6x}\right)\left(\frac{4\sin 3x}{3\sin 4x}\right)\left(\frac{\sin x}{\sin 2x}\right)$

$$= \lim_{x\to 0}\frac{\frac{\sin 9x}{9x}}{\frac{\sin 10x}{10x}}\cdot\frac{\frac{\sin 7x}{7x}}{\frac{\sin 8x}{8x}}\cdot\frac{\frac{\sin 5x}{5x}}{\frac{\sin 6x}{6x}}\cdot\frac{\frac{\sin 3x}{3x}}{\frac{\sin 4x}{4x}}\cdot\frac{\frac{\sin x}{x}}{2\times\frac{\sin 2x}{2x}}$$

$$= 1\cdot 1\cdot 1\cdot 1\cdot\frac{1}{2\times 1} = \frac{1}{2} \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

**3. (c) :** We have, $f(x) = \sqrt{7-3x} + \log_e x$

Let $g(x) = \sqrt{7-3x}$ and $h(x) = \log_e x$

Now $g(x)$ will be defined, when $7 - 3x \geq 0$

*i.e.*, $x \leq \frac{7}{3}$ $\therefore$ domain of $g$ is $-\infty < x \leq \frac{7}{3}$

Also $h(x)$ will be defined when $x > 0$

$\therefore$ domain of $h$ is $0 < x < \infty$

So, domain of $f(x)$ is domain of $g \cap$ domain of $h$

*i.e.* $0 < x \leq \frac{7}{3}$.

**4. (d) :** We have, $f(x) = \frac{1}{2-\cos 3x}$

We know that, $-1 \leq \cos 3x \leq 1$

$\therefore$ $f(x)$ will be minimum when $\cos 3x = -1$ and minimum value of $f(x)$ is $\frac{1}{3}$.

Again $f(x)$ will be maximum when $\cos 3x = 1$ and maximum value of $f(x)$ is 1.

Therefore, the required range is $\frac{1}{3} \leq f(x) \leq 1$ *i.e.* $\left[\frac{1}{3}, 1\right]$

**5. (b) :** Given that $f(x + y) = f(x)f(y)\ \forall\ x, y$ and $f(5) = 2, f'(0) = 3$

$$\text{Now } f'(5) = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h}$$

$$= \lim_{h\to 0}\frac{f(5)\cdot f(h)-f(5)}{h}$$

$$= f(5)\lim_{h\to 0}\frac{f(h)-1}{h}$$

$$= f(5)\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$[\because f(x + y) = f(x)f(y) \quad \therefore f(0 + 0) = f(0)\cdot f(0)$

$\Rightarrow f(0) = [f(0)]^2$

$\Rightarrow f(0) = 1]$

$\therefore$ $f'(5) = f(5) \times f'(0)$

$= 2 \times 3 = 6$

**6. (d) :** $f(x) = \log_{x^2}(\log x) = \frac{\log_e(\log_e x)}{\log_e x^2}$

$$\Rightarrow f(x) = \frac{\log_e(\log_e x)}{2\log_e x}$$

$$\therefore f'(x) = \frac{1}{2}\cdot\frac{\log_e x\cdot\frac{1}{x\log_e x} - \frac{1}{x}\log_e(\log_e x)}{(\log_e x)^2}$$

$$\text{Now } f'(x) \text{ at } x = e = \frac{1}{2}\frac{\left(\frac{1}{e}-\frac{1}{e}\times 0\right)}{(1)^2} = \frac{1}{2e}$$

**7. (b) :** $f(x) = [x]$ is continuous at $x = 1.5$

$\because$ $\lim_{x\to 1.5^-}[x] = 1$ and $\lim_{x\to 1.5^+}[x] = 1$

**8. (c) :** $f(x) = a\sin|x| + be^{|x|}$

$$\therefore f(x) = \begin{cases} a\sin x + be^x, & \text{when } x \geq 0 \\ -a\sin x + be^{-x}, & \text{when } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} a\cos x + be^x, & \text{when } x \geq 0 \\ -a\cos x - be^{-x}, & \text{when } x < 0 \end{cases}$$

Since, at $x = 0$, the function is differentiable.

$\therefore$ $Rf'(0) = Lf'(0)$

$\Rightarrow$ $a + b = -a - b$ $\Rightarrow$ $a + b = 0$

**9. (d) :** $\lim_{x\to 2}\frac{f(4)-f(x^2)}{x-2} = \lim_{x\to 2}\frac{f(4)-f(x^2)}{x^2-4}\cdot(x+2)$

$$= -\lim_{x\to 2}\frac{f(x^2)-f(4)}{x^2-4}\cdot(x+2)$$

$$= -\lim_{h\to 0}\frac{f(h+4)-f(4)}{h}\times\lim_{x\to 2}(x+2)$$

$$\left[\begin{array}{l}\text{Let } x^2-4 = h \quad \therefore x^2 = h+4 \\ x\to 2 \Rightarrow h\to 0\end{array}\right]$$

$= -4f'(4) = -4 \times 5 = -20$

**10. (c) :** $\frac{d}{dx}[x(x-1)(x-2)(x-3)]$

$= \frac{d}{dx}[(x^2-3x)(x^2-3x+2)]$

$= (x^2 - 3x)\cdot(2x - 3) + (x^2 - 3x + 2)(2x - 3)$

$= (2x - 3)(x^2 - 3x + x^2 - 3x + 2)$

$= (2x - 3)(2x^2 - 6x + 2)$

$= 2\ (2x - 3)(x^2 - 3x + 1)$

$\therefore$ At $x = 3$, the value of $\frac{d}{dx}[x(x-1)(x-2)(x-3)]$

$= 2 \times (6 - 3)(9 - 9 + 1)$

$= 2 \times 3 = 6$

**11. (a) :** We have, $-1 \le \sin x \le 1$

$\therefore \quad -1 \le \sin e^{1/x} \le 1$

$\Rightarrow \quad -x \le x \sin e^{1/x} \le x$

Since, $\lim\limits_{x\to 0} -x = \lim\limits_{x\to 0} x = 0$

$\therefore$ By Sandwich Theorem, we get

$$\lim_{x\to 0} x \sin e^{1/x} = 0$$

**12. (d) :** $f(x) = \sin^{-1}\left(\dfrac{1-\cos 2x}{2\sin x}\right)$

$$= \sin^{-1}\left(\frac{2\sin^2 x}{2\sin x}\right) = \sin^{-1}(\pm \sin x)$$

$f(x) = \pm x \Rightarrow f'(x) = \pm 1$

So, $|f'(x)| = 1$

**13. (d) :** Given that, $\sin x = \dfrac{2t}{1+t^2}$

$$\Rightarrow \quad x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$$

$$\Rightarrow \quad x = 2\tan^{-1} t \quad \Rightarrow \quad \frac{dx}{dt} = \frac{2}{1+t^2}$$

Again $\tan y = \dfrac{2t}{1-t^2}$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{2t}{1-t^2}\right) \Rightarrow y = 2\tan^{-1} t$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2}{1+t^2} \quad \therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt}\right) \div \left(\frac{dx}{dt}\right) = 1$$

**14. (a) :** $y = \dfrac{x}{x+1} + \dfrac{x+1}{x} = 1 - \dfrac{1}{x+1} + 1 + \dfrac{1}{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{(x+1)^2} - \frac{1}{x^2} = (x+1)^{-2} - x^{-2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -2(x+1)^{-3} + 2x^{-3} = \frac{-2}{(x+1)^3} + \frac{2}{x^3}$$

$$\therefore \quad \left[\frac{d^2y}{dx^2}\right]_{x=1} = -\frac{2}{8} + 2 = \frac{14}{8} = \frac{7}{4}$$

**15. (a) :** Let $A$ and $r$ be the area and radius of a circle respectively.

$$\because \quad A = \pi r^2 \quad \Rightarrow \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

But it is given that, $\dfrac{dA}{dt} = 3c\dfrac{dr}{dt} \quad \therefore \quad 3c\dfrac{dr}{dt} = 2\pi r \dfrac{dr}{dt}$

Now, when $r = 6$, then $3c = 2\pi \times 6$

$\Rightarrow \quad c = 4\pi$

**16. (b) :** $f(x) = \sin x - kx - c$

$f'(x) = \cos x - k$

Now $f(x)$ will be a decreasing function when $f'(x) \le 0$

*i.e.*, $\cos x \le k$

$\Rightarrow$ for decreasing function, $k \ge 1 [\because \ -1 \le \cos x \le 1]$

**17. (d) :** $y = 3x^2 - 11x - 15$

$$\therefore \quad \frac{dy}{dx} = 6x - 11$$

But $6x - 11 = \dfrac{227-5}{11-5} = \dfrac{222}{6} = 37$

$\Rightarrow \quad 6x = 48 \quad \Rightarrow \quad x = 8$

**18. (c) :** Let $f(x) = x^2 - 8x + 17$

$\Rightarrow \quad f'(x) = 2x - 8$

$\therefore \ f'(x) = 0$ gives $2x - 8 = 0 \quad \Rightarrow \quad x = 4$

Now $f''(x) = 2 > 0$

Thus $f(x)$ is minimum at $x = 4$.

Now, $f(4) = 4^2 - 8 \times 4 + 17 = 33 - 32 = 1$

$\therefore$ The minimum value of $x^2 - 8x + 17$ is 1.

**19. (b) :** Here $y = f(x^2 + 2)$

$$\therefore \quad \frac{dy}{dx} = 2xf'(x^2+2)$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=1} = 2 \times 1 f'(1^2+2) = 2f'(3)$$

$$= 2 \times 5 \qquad [\because \ f'(3) = 5]$$

$$= 10$$

**20. (a) :** $\displaystyle\int \frac{\sin^2 x}{1+\cos x}\,dx = \int \frac{(1+\cos x)(1-\cos x)}{(1+\cos x)}\,dx$

$$= \int (1-\cos x)\,dx = x - \sin x + C$$

**21. (c) :** $\displaystyle\int e^x \left(\frac{1+\sin x}{1+\cos x}\right)dx$

$$= \int e^x \left(\frac{1}{1+\cos x} + \frac{\sin x}{1+\cos x}\right)dx$$

$$= \int e^x \left(\frac{1}{2}\sec^2\frac{x}{2} + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx$$

$$= \int e^x \left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right)dx$$

$$= e^x \tan\frac{x}{2} + C \quad \left[\because \int e^x [f(x) + f'(x)]dx = e^x f(x) + C\right]$$

**22. (c) :**

$$\text{L.H.S.} = \int \frac{\cos 8x + 1}{\tan 2x - \cot 2x}\,dx = \int \frac{2\cos^2 4x \times \sin 2x \cos 2x}{-\cos 4x}\,dx$$

$$= \frac{-1}{2}\int 2\sin 4x\ \cos 4x\ dx$$

$$= -\frac{1}{2}\int \sin 8x\, dx = \frac{1}{16}\cos 8x + C = a\cos 8x + C$$

$$\therefore \quad a = \frac{1}{16}$$

**23. (c) :** Let $I = \int_0^{\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$

$$= \int_0^{\pi} \frac{(\pi - x)\sin^{2n}(\pi - x)}{\sin^{2n}(\pi - x) + \cos^{2n}(\pi - x)}\, dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$$

$$= \pi\int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx - I$$

$$\Rightarrow \quad 2I = \pi \cdot 2\int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$$

$$\Rightarrow \quad I = \pi \cdot \frac{\pi}{4} \quad \left[\because \int_0^{\pi/2} \frac{\sin^{n} x}{\sin^{n} x + \cos^{n} x}\, dx = \frac{\pi}{4}\right]$$

$$\Rightarrow \quad I = \frac{\pi^2}{4}$$

**24. (d) :** Let $I = \int_0^{\pi} x\, f(\sin x)\, dx$

$$= \int_0^{\pi} (\pi - x) f\{\sin(\pi - x)\} dx$$

$$= \pi\int_0^{\pi} f(\sin x) dx - I \quad [\because \ \sin(\pi - x) = \sin x]$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} f(\sin x)\, dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx = \frac{\pi}{2}\cdot 2\int_0^{\pi/2} f(\sin x)\, dx$$

$$\Rightarrow \quad \int_0^{\pi} x\, f(\sin x)\, dx = \pi\int_0^{\pi/2} f(\sin x)\, dx$$

Also, $\int_0^{\pi} xf(\sin x)\, dx = A\int_0^{\pi/2} f(\sin x)\, dx$

$\therefore$ On comparing, we get $A = \pi$

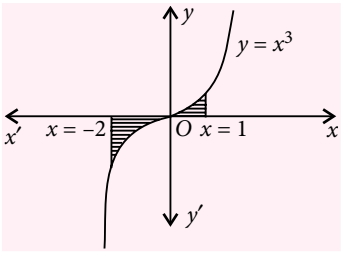
**25. (d) :** Required area

$$= \int_{-2}^{1} x^3\, dx = \left|\int_{-2}^{0} x^3 dx\right| + \int_0^1 x^3 dx$$

$$= \left|\left[\frac{1}{4}x^4\right]_{-2}^{0}\right| + \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{16}{4} + \frac{1}{4}$$

$$= \frac{17}{4} \text{ sq. units}$$




**26. (a) :** We have, $y^2 = 4x$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2}{y}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{2}{2} = 1 \quad \Rightarrow \quad -\left(\frac{dx}{dy}\right)_{(1,2)} = -1$$

Now equation of the normal to the curve at (1, 2) is

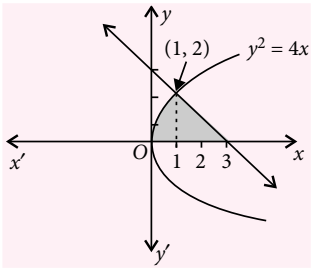
$y - 2 = -1\ (x - 1)$

$\Rightarrow \ y - 2 = -x + 1$

$\Rightarrow \ x + y = 3$

The line $x + y = 3$ meets the $x$-axis at $x = 3$.

$\therefore$ Required area




$$= \int_0^1 \sqrt{4x}\, dx + \int_1^3 (3 - x)\, dx$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^3 = \frac{4}{3}(1) + \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right]$$

$$= \frac{4}{3} + 2 = \frac{10}{3} \text{ sq. units}$$

**27. (a) :** Area bounded by $y = \cos x$, $y = \sin x$ and $y$-axis when $0 \le x \le \frac{\pi}{2}$ is

$$\int_0^{\pi/2} \cos x\, dx - \left[\int_0^{\pi/4} \sin x\, dx + \int_{\pi/4}^{\pi/2} \cos x\, dx\right]$$

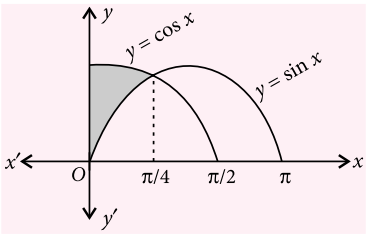
$$= [\sin x]_0^{\pi/2} - \left([-\cos x]_0^{\pi/4} + [\sin x]_{\pi/4}^{\pi/2}\right)$$

$$= 1 - \left[\left(-\frac{1}{\sqrt{2}} + 1\right) + \left(1 - \frac{1}{\sqrt{2}}\right)\right]$$

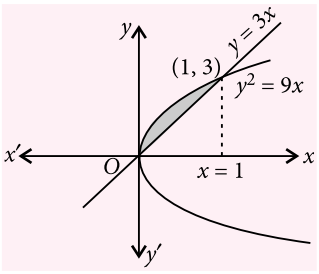
$$= 1 - \left[2\left(1 - \frac{1}{\sqrt{2}}\right)\right]$$

$$= 1 - 2 + \sqrt{2}$$

$$= (\sqrt{2} - 1) \text{ sq. units}$$



**28. (c) :** The area bounded by the curve $y^2 = 9x$ and $y = 3x$ is




$$\int_0^1 \sqrt{9x}\,dx - \int_0^1 3x\,dx$$

$$= \left[\frac{3x^{3/2}}{\frac{3}{2}}\right]_0^1 - \left[\frac{3x^2}{2}\right]_0^1$$

$$= \left(3\times\frac{2}{3}\right) - \frac{3}{2} = \frac{1}{2} \text{ sq. units}$$

**29. (a) :** We have, $y'(y^2 - x) = y$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{y^2 - x} \Rightarrow \frac{dx}{dy} = y - \frac{1}{y}x$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{y}x = y \quad \therefore \quad \text{I.F.} = e^{\int \frac{1}{y}dy} = y$$

$\therefore$ Required solution is $x \cdot y = \int y^2 dy$

$$\Rightarrow xy = \frac{1}{3}y^3 + C' \Rightarrow 3xy = y^3 + 3C'$$

$\Rightarrow y^3 - 3xy = C$

**30. (c) :** The given differential equation is

$$\left[2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^3y}{dx^3}$$

$$\Rightarrow \left[2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^3y}{dx^3}\right)^2$$

$\therefore$ Order and degree of the differential equation is 3 and 2 respectively.

**31. (a) :** Let $I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)}\,dx$

$$= \int_2^4 \frac{2\log x}{2\log x + 2\log(6-x)}\,dx = \int_2^4 \frac{\log x}{\log x + \log(6-x)}\,dx$$

$$\Rightarrow I = \int_2^4 \frac{\log(6-x)}{\log(6-x) + \log x}\,dx \left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx\right]$$

$$\therefore \quad 2I = \int_2^4 dx = 2 \quad \therefore \quad I = 1$$

**32. (b) :** Let $I = \int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^5\left(1 + \frac{1}{x^4}\right)^{3/4}}$

Put $1 + \frac{1}{x^4} = t \Rightarrow \frac{-4}{x^5}dx = dt$

$$\therefore \quad I = -\frac{1}{4}\int \frac{1}{t^{3/4}}\,dt$$

$$= -\frac{1}{4}\cdot\frac{t^{1/4}}{\frac{1}{4}} + C = -t^{1/4} + C = -\left(1 + \frac{1}{x^4}\right)^{1/4} + C$$

**33. (c) :** Here $g(x) = \begin{cases} k\sqrt{x+1} & , \quad 0 \le x \le 3 \\ mx + 2 & , \quad 3 < x \le 5 \end{cases}$

is differentiable at $x = 3$.

$\therefore$ It must be continuous at $x = 3$.

$$\therefore \lim_{x\to 3^+} g(x) = \lim_{x\to 3^-} g(x) \Rightarrow \lim_{x\to 3} mx + 2 = \lim_{x\to 3} k\sqrt{x+1}$$

$$\Rightarrow 3m + 2 = 2k \qquad \ldots(i)$$

Again $Rg'(3) = \lim_{h\to 0} \frac{g(3+h) - g(3)}{h}$

$$= \lim_{h\to 0} \frac{m(3+h) + 2 - 2k}{h} = \lim_{h\to 0} \frac{3m + mh + 2 - 2k}{h}$$

$$= \lim_{h\to 0} \frac{(3m+2) + mh - 2k}{h} = \lim_{h\to 0} \frac{mh}{h} = m \quad [\text{from (i)}]$$

Also, $Lg'(3) = \lim_{h\to 0} \frac{g(3-h) - g(3)}{-h}$

$$= \lim_{h\to 0} \frac{k\sqrt{4-h} - 2k}{-h} = \lim_{h\to 0} \frac{k(\sqrt{4-h} - 2)}{-h}$$

$$= \lim_{h\to 0} \frac{k[(4-h) - 4]}{-h(\sqrt{4-h} + 2)} = \lim_{h\to 0} \frac{k}{\sqrt{4-h} + 2} = \frac{k}{4}$$

Since g($x$) is differentiable at $x = 3$

$$\therefore \quad m = \frac{k}{4} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$m = \frac{2}{5},\ k = \frac{8}{5} \qquad \therefore \quad m + k = \frac{10}{5} = 2$$

**34. (b) :** Let $I = \int_0^1 \frac{1}{(x^2+16)(x^2+25)}\,dx$

$$= \frac{1}{9}\int_0^1 \frac{(x^2+25) - (x^2+16)}{(x^2+16)(x^2+25)}\,dx$$

$$= \frac{1}{9}\left[\int_0^1 \frac{dx}{x^2+4^2} - \int_0^1 \frac{dx}{x^2+5^2}\right]$$

$$= \frac{1}{9}\left[\frac{1}{4}\tan^{-1}\frac{x}{4}\Big|_0^1 - \frac{1}{5}\tan^{-1}\frac{x}{5}\Big|_0^1\right]$$

$$= \frac{1}{9}\left[\frac{1}{4}\tan^{-1}\left(\frac{1}{4}\right) - \frac{1}{5}\tan^{-1}\left(\frac{1}{5}\right)\right]$$

**35. (b) :** $f(x) = \log(\sin x)$ is continuous in $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ and differentiable in $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ as its derivative.

Now, $f'(x) = \frac{1}{\sin x} \times \cos x = \cot x$

$\therefore \quad f\left(\frac{\pi}{6}\right) = \log\left(\sin\frac{\pi}{6}\right) = \log\frac{1}{2}$

$f\left(\frac{5\pi}{6}\right) = \log\left(\sin\left(\frac{5\pi}{6}\right)\right) = \log\left(\sin\left(\pi - \frac{\pi}{6}\right)\right) = \log\frac{1}{2}$

$\therefore$ There exists a point $c \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ such that

$$f'(c) = \frac{f\left(\frac{\pi}{6}\right) - f\left(\frac{5\pi}{6}\right)}{\frac{\pi}{6} - \frac{5\pi}{6}}$$

$\Rightarrow \quad \cot c = 0 \quad \Rightarrow \quad c = \frac{\pi}{2}$

**36. (b, c) :** The equation of the circle is

$(x - \alpha)^2 + (y - \alpha)^2 = r^2$

$\Rightarrow \quad x^2 + y^2 - 2\alpha x - 2\alpha y + 2\alpha^2 - r^2 = 0$ ...(i)

$\Rightarrow \quad 2x + 2yy' - 2\alpha - 2\alpha y' = 0$ ...(ii)

$\therefore \quad \alpha = \frac{x + yy'}{1 + y'}$ ...(iii)

Differentiating (ii), we get $1 + y'^2 + yy'' - \alpha y'' = 0$

$$\Rightarrow \quad 1 + y'^2 + yy'' - \frac{x + yy'}{(1 + y')} y'' = 0 \text{ [from (iii)]}$$

$\Rightarrow \quad (y - x)y'' + (1 + y' + y'^2)y' + 1 = 0$

Now comparing with the given differential equation, we get $P = y - x$, $Q = (1 + y' + y'^2)$

**37. (a,b, c) :** $f(x + y) = f(x) f(a - y) + f(y) f(a - x)$ ...(i)

Putting $x = y = 0$, we get

$f(0) = f(0)f(a) + f(0)f(a)$

$\Rightarrow \quad f(0) = 2f(0)f(a) \quad \Rightarrow \quad f(a) = \frac{1}{2}$

Again putting $y = 0$ in (i), we get

$f(x) = f(x)f(a) + f(0)f(a - x)$

$\Rightarrow \quad f(x) = f(a - x)$

Now putting $y = a - x$ in (i), we get

$f(a) = (f(x))^2 + (f(a - x))^2$

$\Rightarrow \quad f(x) = \pm\frac{1}{2} \quad \therefore \quad f(x) = \frac{1}{2} \qquad \left[\text{As} \quad f(x) \neq -\frac{1}{2}\right]$

**38. (c) :** $f'(x) = 6x^2 - 18ax + 12a^2 = 0$ (for max or min)

$\Rightarrow \quad x = a, 2a$

Now, it is given that $x_2 = x_1^2$

$\Rightarrow \quad a$ cannot be negative

If $a > 0$, then local minimum occurs at $x = 2a$ and local maximum at $x = a$. Thus, $x_1 = a$ and $x_2 = 2a$

$\Rightarrow \quad 2a = a^2$

$\Rightarrow \quad a = 2$ (if $a = 0$, $f(x)$ is always increasing)

**39. (a) :** $|\sin x| + |\cos x| = \sqrt{(|\sin x| + |\cos x|)^2}$

$= \sqrt{\sin^2 x + \cos^2 x + 2|\sin x||\cos x|}$

$= \sqrt{1 + |2\sin x \cos x|} = \sqrt{1 + |\sin 2x|}$

Now $0 \le |\sin 2x| \le 1$

$\Rightarrow \quad 1 \le 1 + |\sin 2x| \le 2 \quad \Rightarrow \quad 1 \le \sqrt{1 + |\sin 2x|} \le \sqrt{2}$

$\therefore \quad [|\sin x| + |\cos x|] = 1$

Now $y = [|\sin x| + |\cos x|] \quad \therefore \quad y = 1$ ...(i)

and $x^2 + y^2 = 10$ ...(ii)

Points of intersection of (i) and (ii) are (3, 1), (–3, 1)

Now $x^2 + y^2 = 10$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(3,1)} = -3 \text{ and } \left.\frac{dy}{dx}\right|_{(-3,1)} = 3$$

Now slope of (i) is $\frac{dy}{dx} = 0$

$\therefore$ Angle between the two curves at (3, 1) is

$$\tan^{-1}\left|\frac{-3 - 0}{1 + (-3 \times 0)}\right| = \tan^{-1} 3$$

and angle between the two curves at (–3, 1) is

$$\tan^{-1}\left|\frac{3 - 0}{1 + (3 \times 0)}\right| = \tan^{-1} 3$$

**40. (a, d) :** $g(x) = \int_0^x \cos 4t \, dt = \left.\frac{\sin 4t}{4}\right|_0^x$

$\Rightarrow \quad g(x) = \frac{1}{4}\sin 4x$ ...(i)

$\therefore \quad g(x + \pi) = \frac{1}{4}\sin 4(x + \pi) = \frac{1}{4}\sin(4x + 4\pi)$

$= \frac{1}{4}\sin 4x$

$\therefore \quad g(x + \pi) = g(x) = \frac{1}{4}\sin 4x$ ...(ii)

Now putting $x = 0$ in (ii), we get

$g(\pi) = g(0) = 0$

$\therefore \quad g(x + \pi) = g(x) \pm 0 = g(x) \pm g(\pi)$

# ACE YOUR WAY CBSE XII

Series 5

## APPLICATION OF DERIVATIVES

### RATE OF CHANGE OF QUANTITIES

Let $y = f(x)$ be a relation between two variables $x$ and $y$. Then $\frac{dy}{dx}$(or $f'(x)$) represents the rate of change of $y$ with respect to $x$ and $\left.\frac{dy}{dx}\right|_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of $y$ with respect to $x$ at $x = x_0$.

Further, if two variables $x$ and $y$ vary with respect to another variable $t$ *i.e.*, $x$ and $y$ are functions of $t$, then by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} \quad \text{if } \frac{dx}{dt} \neq 0$$

Thus, the rate of change of $y$ w.r.t $x$ can be calculated using the rate of change of $y$ and $x$ w.r.t. $t$.

**Note:**

$\frac{dy}{dx}$ is positive if $y$ increases as $x$ increases and is negative if $y$ decreases as $x$ increases.

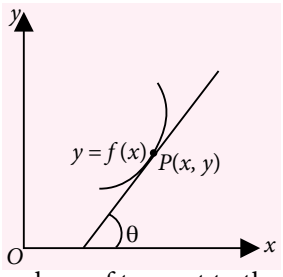
### INCREASING AND DECREASING FUNCTIONS

| | |
|---|---|
| **Increasing Function** | A function $f(x)$ defined on $]a, b[$ is said to be increasing<br>(i) If $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in ]a, b[$<br>or $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in ]a, b[$.<br>OR<br>(ii) If $f'(x) \geq 0$ for each $x \in ]a, b[$. |
| **Strictly Increasing Function** | A function $f(x)$ defined on $]a, b[$ is said to be strictly increasing<br>(i) If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in ]a, b[$<br>or $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in ]a, b[$.<br>OR<br>(ii) If $f'(x) > 0$ for each $x \in ]a, b[$. |
| **Decreasing Function** | A function $f(x)$ defined on $]a, b[$ is said to be decreasing<br>(i) If $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in ]a, b[$<br>or $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in ]a, b[$.<br>OR<br>(ii) If $f'(x) \leq 0$ for each $x \in ]a, b[$. |
| **Strictly Decreasing Function** | A function $f(x)$ defined on $]a, b[$ is said to be strictly decreasing<br>(i) If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in ]a, b[$<br>or $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in ]a, b[$.<br>OR<br>(ii) If $f'(x) < 0$ for each $x \in ]a, b[$. |

## TANGENTS AND NORMALS

### Slope of a line

Consider a curve $y = f(x)$ and a point $P(x, y)$ on this curve as shown in Fig. If tangent to the curve at $P(x, y)$ makes an angle $\theta$ with the positive direction of $x$-axis, then, at the point $P(x, y)$:




$\frac{dy}{dx} = \tan\theta = m$ = gradient or slope of tangent to the curve at $P(x, y)$.

### Equation of Tangent

The equation of tangent to the curve $y = f(x)$ at a given point $P(x_1, y_1)$ is given by

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

### Equation of Normal

Slope of normal at $(x_1, y_1)$

$$= \frac{-1}{\text{Slope of tangent at } (x_1, y_1)} = \frac{-1}{(dy/dx)_{(x_1, y_1)}}$$

$\therefore$ The equation of normal to the curve $y = f(x)$ at a given point $P(x_1, y_1)$ is given by

$$y - y_1 = \frac{-1}{(dy/dx)_{(x_1, y_1)}} (x - x_1)$$

### Particular cases:

**Case I :** If $\frac{dy}{dx} = 0$ at $(x_1, y_1)$, then tangent is parallel to $x$-axis. In this case, the equation of the tangent at the point $(x_1, y_1)$ is given by $y = y_1$.

**Case II :** If $\frac{dy}{dx} = \infty$ at $(x_1, y_1)$, then tangent is perpendicular to $x$-axis. In this case, the equation of the tangent at the point $(x_1, y_1)$ is given by $x = x_1$.

## ERROR AND APPROXIMATIONS

Let $y = f(x)$ and $\Delta x$ be the small change in $x$ and $\Delta y$ be the small change in $y$. Then $\Delta y = \frac{dy}{dx}(\Delta x)$ is used in the calculation of small changes (or errors).

*Steps to find the approximate value of certain quantity* :

1. Find $\Delta x$ and $x$
2. $\Delta y = \frac{dy}{dx}(\Delta x)$
3. $\Delta y = f(x + \Delta x) - f(x)$ or $f(x + \Delta x) = \Delta y + f(x)$

## MAXIMA AND MINIMA

Let $f$ be a function defined on an interval $I$. Then $f$ is said to have

(a) A maximum value in $I$, if there exists a point $c$ in $I$ such that $f(c) > f(x)$, for all $x \in I$.
In this case, $f(c)$ is called the maximum value of $f$ in $I$ and the point $c$ is called a point of maximum value of $f$ in $I$.

(b) A minimum value in $I$, if there exists a point $c$ in $I$ such that $f(c) < f(x)$, for all $x \in I$.
In this case, $f(c)$ is called the minimum value of $f$ in $I$ and the point $c$ is called a point of minimum value of $f$ in $I$.

(c) An extreme value in $I$ if there exists a point $c$ in $I$ such that $f(c)$ is either a maximum value or a minimum value of $f$ in $I$.
In this case, $f(c)$ is called an extreme value of $f$ in $I$ and the point $c$ is called an extreme point.

### First Derivative Test

Let $f$ be a function defined on an open interval $I$ and $f$ be continuous at a critical point $c$ in $I$. Then

(i) If $f'(x)$ changes sign from positive to negative as $x$ increases through $c$ *i.e.* if $f'(x) > 0$ at every point sufficiently close to and to the left of $c$, and $f'(x) < 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a point of local maxima.

(ii) If $f'(x)$ changes sign from negative to positive as $x$ increases through $c$ *i.e.* if $f'(x) < 0$ at every point sufficiently close to and to the left of $c$, and $f'(x) > 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a point of local minima.

(iii) If $f'(x)$ does not change sign as $x$ increases through $c$, then $c$ is neither a point of local maxima nor a point of local minima. Infact, such a point is called point of inflection.

### Second Derivative Test

Let $f$ be a function defined on an open interval $I$ and let $c \in I$. Let $f$ be twice differentiable at $c$. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$
The value $f(c)$ is local maximum value of $f$.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
In this case, $f(c)$ is local minimum value of $f$.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

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## GREATEST AND LEAST VALUES OF A FUNCTION

Let $f(x)$ be a function defined on the interval $[a, b]$

1. Greatest value or absolute maximum or global maximum value of a function :
   $f(c)$ is said to be the greatest value of $f(x)$ in an interval $[a, b]$ if
   $f(c) \geq f(x)$ for all $x \in [a, b]$
2. Least value or absolute minimum or global minimum value of a function:
   $f(c)$ is said to be the least value of $f(x)$ in an interval $[a, b]$ if
   $f(c) \leq f(x)$ for all $x \in [a, b]$
3. $f(c)$ is said to be the greatest value or least value of $f(x)$ in an interval $(a, b)$ respectively if $f(c) \geq f(x)$ for all $x, \in ]a, b[$ or $f(c) \leq f(x)$ for all $x \in ]a, b[$.

**Note:**

- Every monotonic function assumes its maximum/ minimum value at the end points of its domain.
- Every continuous function on a close interval has a maximum and minimum.

### Very Short Answer Type

1. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x + 3)$. Determine the rate of change of volume with respect to $x$.
2. Show that $f(x) = e^{1/x}$ is a strictly decreasing function for all $x > 0$.
3. Find the slope of the curve $y^2 = x$ at point $x = 1$.
4. If the radius of a sphere is measured as 9 m with an error of 0.03 m, then find the approximate error in calculating its surface area.
5. Prove that the function, $f(x) = \log x$ do not have maxima or minima.

### Short Answer Type

6. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the $y$-coordinate is changing 8 times as fast as the $x$-coordinate.
7. Find the equations of tangent and normal to the curve $y = x^2 + 4x + 1$ at the point whose abscissa is 3.
8. For the function $f(x) = -2x^3 - 9x^2 - 12x + 1$, find the interval(s) in which
   (i) $f(x)$ is decreasing
   (ii) $f(x)$ is increasing.
9. Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$.
10. Find the local maxima or local minima of $f(x) = x^3 - 6x^2 + 9x + 15$. Also, find the local maximum or local minimum values as the case may be.

### Long Answer Type

11. Show that the semivertical angle of a right circular cone of given surface area and maximum volume is $\sin^{-1}(1/3)$.
12. Find the values of $x$ for which the function $f(x) = x^x, x > 0$ is (a) increasing (b) decreasing.
13. Find the maximum and minimum values of $\sec x + \log \cos^2 x$.
14. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).
15. (i) If $f(x) = a \log |x| + bx^2 + x$ has extreme values at $x = -1$ and at $x = 2$, then find $a$ and $b$.
    (ii) Show that $f(x) = \cos(2x + \pi/4)$ is an increasing function on $(3\pi/8, 7\pi/8)$.

### SOLUTIONS

1. Let $V$ be the volume of the balloon. Then

$$V = \frac{4}{3}\pi r^3 = \frac{4\pi}{3}\left\{\frac{3}{4}(2x+3)\right\}^3 = \frac{9\pi}{16}.(2x+3)^3$$

$$\Rightarrow \frac{dV}{dx} = \frac{9\pi}{16}.3(2x+3)^2\frac{d}{dx}(2x+3)$$

$$\Rightarrow \frac{dV}{dx} = \frac{27\pi}{8}(2x+3)^2.$$

2. $f(x) = e^{1/x} \Rightarrow f'(x) = -\frac{1}{x^2}e^{1/x} < 0$ for all $x > 0$

$[\because x^2 > 0$ and $e^{1/x} > 0 \ \forall \ x > 0]$.

Thus, $f'(x) < 0$ for all $x > 0$.

Hence, $f(x)$ is strictly decreasing function for all $x > 0$.

3. Given equation of curve is $y^2 = x$ ...(i)

When $x = 1$, $y^2 = 1 \Rightarrow y = \pm 1$

$\therefore$ Points are (1, 1) and (1, –1).

Differentiating both sides of (i) w.r.t. $x$, we get

$$2y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Now, $\left(\frac{dy}{dx}\right)_{(1,\,1)} = \frac{1}{2}$ and $\left(\frac{dy}{dx}\right)_{(1,\,-1)} = -\frac{1}{2}$

Hence, slope of tangents at points (1, 1) and (1, –1) are $\frac{1}{2}$ and $-\frac{1}{2}$ respectively.

**4.** Radius of sphere($r$) = 9 m
Error in measurement ($\Delta r$) = 0.03 m
Surface area of sphere($S$) = $4\pi r^2$

$$\therefore \quad \Delta S = \frac{dS}{dr} \times \Delta r = 8\pi r \times \Delta r$$
$$= 8\pi \times 9 \times 0.03 = 2.16\ \pi\ m^2$$

**5.** Given that, $f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$

Now, $f'(x) = 0 \Rightarrow \frac{1}{x} = 0$, which is not possible for any real $x$.
Hence $f(x) = \log x$ do not have maxima or minima.

**6.** Given equation of the curve is

$$6y = x^3 + 2$$

Differentiating both sides w.r.t. $t$, we get

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} \Rightarrow 6 \cdot 8\frac{dx}{dt} = 3x^2\frac{dx}{dt}$$
$$\left[\because \frac{dy}{dt} = 8 \cdot \frac{dx}{dt} \text{ (Given)}\right]$$

$\Rightarrow 48 = 3x^2 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

When $x = 4$, $y = \frac{(4)^3 + 2}{6} = \frac{66}{6} = 11$

When $x = -4$, $y = \frac{(-4)^3 + 2}{6} = \frac{-62}{6} = -\frac{31}{3}$

Hence, the required points are $(4, 11)$ and $\left(-4, -\frac{31}{3}\right)$.

**7.** Given equation of curve is $y = x^2 + 4x + 1$. ...(i)
The ordinate of the point whose abscissa is 3 is given by

$$y = (3)^2 + 4(3) + 1 = 22$$

So, the given point on the curve is (3, 22).
Differentiating both sides of (i) w.r.t. $x$, we get

$$\frac{dy}{dx} = 2x + 4$$

Slope of tangent at (3, 22) = $\left.\frac{dy}{dx}\right|_{(3,\ 22)} = 2(3) + 4 = 10$

Equation of tangent at (3, 22) is

$$y - 22 = 10(x - 3) \Rightarrow 10x - y - 8 = 0$$

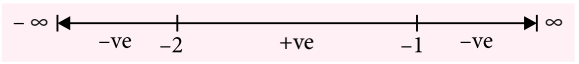
Slope of normal at (3, 22) = $\frac{-1}{\left.\frac{dy}{dx}\right|_{(3,22)}} = -\frac{1}{10}$

$\therefore$ Equation of normal at (3, 22) is

$$y - 22 = -\frac{1}{10}(x - 3)$$

$\Rightarrow 10(y - 22) = -(x - 3) \Rightarrow x + 10y = 223$

**8.** Given that, $f(x) = -2x^3 - 9x^2 - 12x + 1$
$\therefore f'(x) = -6x^2 - 18x - 12 = -6(x^2 + 3x + 2)$
Now $f'(x) = 0 \Rightarrow x^2 + 3x + 2 = 0 \Rightarrow x = -1, -2$
Sign scheme for $f'(x)$ is




From sign scheme for $f'(x)$, it is clear that
(i) $f(x)$ is decreasing in $(-\infty, -2)$ and $(-1, \infty)$
Since $f(x)$ is continuous at $x = -2$ and $x = -1$, therefore, it is also decreasing in $(-\infty, -2]$ and $[-1, \infty)$.
(ii) $f(x)$ is increasing in $(-2, -1)$. Since $f(x)$ is continuous at $x = -2$ and $x = -1$, therefore, it is also increasing in $[-2, -1]$.

**9.** Given that, $f(x) = x^3 - 7x^2 + 15$
$\therefore f'(x) = 3x^2 - 14x$
Let $\delta x$ be the small change in $x$
Then, $f(x + \delta x) = f(x) + \delta x \cdot f'(x)$
$= x^3 - 7x^2 + 15 + \delta x\,(3x^2 - 14x)$ ...(i)
Putting $x = 5$ and $x + \delta x = 5.001$ so that $\delta x = 0.001$ in (i), we get

$$f(5.001) = 5^3 - 7 \cdot 5^2 + 15 + (0.001)(3 \times 5^2 - 14 \times 5)$$
$$= 125 - 175 + 15 + 0.001(5)$$
$$= -35 + 0.005 = -34.995$$

**10.** Given that, $f(x) = x^3 - 6x^2 + 9x + 15$
$\Rightarrow f'(x) = 3x^2 - 12x + 9$.
For local maxima or minima, we must have $f'(x) = 0$.
Now, $f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$
$\Rightarrow 3(x - 3)(x - 1) = 0 \Rightarrow x = 3$ or $x = 1$

**Case I :**
When $x = 3$
In this case, when $x$ is slightly less than 3 then $f'(x)$ is negative and when $x$ is slightly more than 3 then $f'(x)$ is positive.
Thus, $f'(x)$ changes sign from negative to positive as $x$ increases through 3.
So, $x = 3$ is a point of local minima.
$\therefore$ Local minimum value = $f(3) = 15$

**Case II :**
When $x = 1$
In this case, when $x$ is slightly less than 1 then $f'(x)$ is positive and when $x$ is slightly more than 1 then $f'(x)$ is negative.

Thus, $f'(x)$ changes sign from positive to negative as $x$ increases through 1.

So, $x = 1$ is a point of local maxima.

$\therefore$ Local maximum value $= f(1) = 19$.

**11.** Let $r$ be the radius, $l$ be the slant height and $h$ be the height of the cone. Let $S$ denote the surface area and $V$ denote the volume of the cone.

Then, $S = (\pi r^2 + \pi rl) =$ constant

$$\therefore \quad l = \left(\frac{S}{\pi r} - r\right) \qquad ...(i)$$

Now, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2}$

$$\therefore \quad V^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2) = \frac{1}{9}\pi^2 r^4 . \left\{\left(\frac{S}{\pi r} - r\right)^2 - r^2\right\}$$

[Using (i)]

$$= \frac{1}{9}S(Sr^2 - 2\pi r^4).$$

$$\Rightarrow \quad V^2 = \frac{1}{9}S^2 r^2 - \frac{2\pi}{9}Sr^4$$

On differentiating both sides w.r.t. $r$, we get

$$\therefore \quad 2V . \frac{dV}{dr} = \left(\frac{2}{9}S^2 r - \frac{8\pi S}{9}r^3\right) = \frac{2rS}{9}(S - 4\pi r^2) \quad ...(ii)$$

Now, $\frac{dV}{dr} = 0 \Rightarrow r = 0$ or $(S - 4\pi r^2) = 0$

$$\Rightarrow \quad r^2 = \frac{S}{4\pi} \text{ (neglecting } r = 0).$$

Again differentiating (ii) w.r.t. $r$, we get

$$2\left(\frac{dV}{dr}\right)^2 + 2V . \frac{d^2V}{dr^2} = \frac{1}{9}S(2S - 24\pi r^2).$$

Putting $\frac{dV}{dr} = 0$ and $r^2 = \frac{S}{4\pi}$, we get

$$2V . \frac{d^2V}{dr^2} = \frac{1}{9}S(2S - 6S) = -\frac{4}{9}S^2 < 0$$

$\therefore$ When the volume is maximum, we have

$$r^2 = \frac{S}{4\pi} = \frac{(\pi r^2 + \pi rl)}{4\pi} \Rightarrow l = 3r.$$

Now, if $\alpha$ is the semivertical angle of the cone then

$$\frac{r}{l} = \sin\alpha \Rightarrow \frac{r}{3r} = \sin\alpha$$

$$\Rightarrow \sin\alpha = \frac{1}{3} \Rightarrow \alpha = \sin^{-1}\left(\frac{1}{3}\right).$$

Hence, the semivertical angle of a right cone of a given surface and maximum volume is $\sin^{-1}(1/3)$.

**12.** Given, $f(x) = x^x$

or $f(x) = e^{x \log x}$

$$\Rightarrow \quad f'(x) = e^{x\log x} . \frac{d}{dx}(x \log x) = x^x (1 + \log_e x) \quad ...(i)$$

(a) $f(x)$ is increasing

$\Rightarrow f'(x) \geq 0$

$\Rightarrow x^x(1 + \log_e x) \geq 0$ [From (i)]

$\Rightarrow (1 + \log_e x) \geq 0$ [$\because x^x > 0$ when $x > 0$]

$\Rightarrow \log_e x \geq -1$

$\Rightarrow x \geq e^{-1}$

$\Rightarrow x \in \left[\frac{1}{e}, \infty\right[$

$\therefore$ $f(x)$ is increasing on $\left[\frac{1}{e}, \infty\right[$.

(b) $f(x)$ is decreasing

$\Rightarrow f'(x) \leq 0$

$\Rightarrow x^x(1 + \log_e x) \leq 0$

$\Rightarrow (1 + \log_e x) \leq 0$ [$\because x^x > 0$]

$\Rightarrow \log_e x \leq -1$

$\Rightarrow x \leq e^{-1}$

$\Rightarrow 0 < x \leq \frac{1}{e}$ [$\because x > 0$]

$\Rightarrow x \in \left]0, \frac{1}{e}\right]$

$\therefore$ $f(x)$ is decreasing on $\left]0, \frac{1}{e}\right]$.

Hence, $f(x)$ is increasing on $\left[\frac{1}{e}, \infty\right[$ and decreasing on $\left]0, \frac{1}{e}\right]$.

**13.** Let $y = \sec x + \log\cos^2 x$ ...(i)

$$\Rightarrow \quad \frac{dy}{dx} = \sec x \cdot \tan x + \frac{1}{\cos^2 x} 2\cos x(-\sin x)$$

$$= \sec x \cdot \tan x - 2\tan x = \tan x(\sec x - 2) \quad ...(ii)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 x(\sec x - 2) + \tan x(\sec x \cdot \tan x)$$

$$= \sec^3 x + \sec x \cdot \tan^2 x - 2\sec^2 x$$

$$= \sec x(\sec^2 x + \tan^2 x - 2\sec x)$$

For maximum or minimum values of $y$,

put $\frac{dy}{dx} = 0$

$\therefore$ $\tan x(\sec x - 2) = 0$

$\Rightarrow$ $\tan x = 0$, or $\sec x = 2$

$\Rightarrow$ $\tan x = 0$ or $\cos x = \frac{1}{2}$

$\therefore$ $x = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$ [Between 0 and $2\pi$]

At $x=0, \frac{d^2y}{dx^2}=\sec 0(\sec^2 0+\tan^2 0-2\sec 0)$

$= 1(1^2 + 0 - 2) = -1 < 0$

Hence at $x = 0$, $y$ has maximum value.

$\therefore$ Corresponding maximum value of $y =$

$\sec 0 + \log \cos^2 0 = 1 + \log 1 = 1 + 0 = 1$

At $x=\frac{\pi}{3}, \frac{d^2y}{dx^2}=\sec\frac{\pi}{3}\left(\sec^2\frac{\pi}{3}+\tan^2\frac{\pi}{3}-2\sec\frac{\pi}{3}\right)$

$=2\{2^2+(\sqrt{3})^2-2\times 2\}=2\times 3=6>0$

Hence at $x=\frac{\pi}{3}$, $y$ has minimum value.

$\therefore$ Corresponding minimum value of $y =$

$$\sec\frac{\pi}{3}+\log\cos^2\frac{\pi}{3}=2+\log\left(\frac{1}{2}\right)^2$$

$= 2 - 2\log 2 = 2(1 - \log 2)$

At $x=\pi, \frac{d^2y}{dx^2}=\sec\pi(\sec^2\pi+\tan^2\pi-2\sec\pi)$

$= -1(1 + 0 + 2) = -3 < 0.$

Hence at $x = \pi$, $y$ has maximum value.

$\therefore$ Corresponding maximum value of $y =$

$\sec \pi + \log \cos^2 \pi = -1 + \log 1 = -1$

At $x=\frac{5\pi}{3}, \frac{d^2y}{dx^2}=\sec\frac{5\pi}{3}\left(\sec^2\frac{5\pi}{3}+\tan^2\frac{5\pi}{3}-2\sec\frac{5\pi}{3}\right)$

$= 2(4 + 3 - 2 \times 2) = 6 > 0$

Hence at $x=\frac{5\pi}{3}$, $y$ has minimum value

$\therefore$ Corresponding minimum value of $y$ is

$$\sec\frac{5\pi}{3}+\log\cos^2\frac{5\pi}{3}=2+\log\left(\frac{1}{4}\right)$$

$= 2 + (\log 1 - 2\log_e 2) = 2\left(1-\log_e 2\right)$

**14.** Given equation of curve is $x^2 = 4y$ ...(i)

Differentiating w.r.t. $x$, we get

$$2x=4\frac{dy}{dx}$$

$\therefore \quad \frac{dy}{dx}=\frac{x}{2}$

At $(\alpha, \beta)$, $\frac{dy}{dx}=\frac{\alpha}{2}$

$\therefore$ Slope of normal at $(\alpha, \beta) = -\frac{1}{\left.\frac{dy}{dx}\right|_{(\alpha,\beta)}}=-\frac{2}{\alpha}$

Now, equation of normal at $(\alpha, \beta)$ is

$$y-\beta=-\frac{2}{\alpha}(x-\alpha) \qquad ...(*)$$

Since, the normal passes through (1, 2)

$\therefore \quad 2-\beta=-\frac{2}{\alpha}(1-\alpha) \Rightarrow 2\alpha-\alpha\beta=-2+2\alpha$

$\Rightarrow \quad \alpha\beta = 2$ ...(ii)

Again, since $(\alpha, \beta)$ lies on curve (i)

$\therefore \quad \alpha^2 = 4\beta$ ...(iii)

Putting the value of $\beta$ from (ii) in (iii), we get

$$\alpha^2=4\times\frac{2}{\alpha} \Rightarrow \alpha^3=8 \Rightarrow \alpha=2$$

From (ii), $\beta=\frac{2}{\alpha}=\frac{2}{2}=1$

Now from (*), required equation of normal is

$$y-1=-\frac{2}{2}(x-2) \quad i.e., \quad x+y-3=0$$

**15.** (i) Clearly domain $(f) = R - \{0\}$

We have,

$f(x) = a \log |x| + bx^2 + x$

$\Rightarrow \quad f'(x) = \frac{a}{x}+2bx+1$

Since $f(x)$ has extreme values at $x = -1$ and $x = 2$. Therefore,

$f'(-1) = 0$ and $f'(2) = 0$

$\Rightarrow \quad -a - 2b + 1 = 0$ and $\frac{a}{2}+4b+1=0$

$\Rightarrow \quad a + 2b = 1$ ...(i)

and $a + 8b = -2$ ...(ii)

Solving equations (i) and (ii), we get

$$a=2 \quad \text{and} \quad b=-\frac{1}{2}.$$

(ii) We have,

$f(x) = \cos(2x + \pi/4)$

$\Rightarrow \quad f'(x) = -2 \sin (2x + \pi/4)$

Now,

$x \in (3\pi/8, 7\pi/8)$

$\Rightarrow \quad 3\pi/8 < x < 7\pi/8$

$\Rightarrow \quad 3\pi/4 < 2x < 7\pi/4$

$\Rightarrow \quad \pi/4 + 3\pi/4 < 2x + \pi/4 < 7\pi/4 + \pi/4$

$\Rightarrow \quad \pi < 2x + \pi/4 < 2\pi$

$\Rightarrow \quad \sin (2x + \pi/4) < 0$

[$\because$ sine function is negative in third and fourth quadrant]

$\Rightarrow \quad -2 \sin (2x + \pi/4) > 0$

$\Rightarrow \quad f'(x) > 0$

Hence, $f(x)$ is increasing on $(3\pi/8, 7\pi/8)$.

# MATHS MUSING

## SOLUTION SET-153

**1.** **(d) :** The number of ways of 4 letters going to correct destinations is $\binom{8}{4}$. The number of ways of 4 letters going to wrong places is 4! $\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)=9$

$\therefore$ Required probability $= \frac{\binom{8}{4}\times 9}{8!} = \frac{1}{64}$

**2.** **(d) :** The image of $z$ in the $x$-axis is $\bar{z}$ and arg $\bar{z}$ = –arg $z$.

The image of $\arg\left(\frac{\bar{z}-3}{\bar{z}-i}\right)=\frac{\pi}{6}$

$\arg\left(\frac{\bar{z}-3}{\bar{z}-i}\right)=\frac{\pi}{6} \Rightarrow \arg\left(\frac{z-3}{z+i}\right)=\frac{-\pi}{6}$

$\Rightarrow \arg\left(\frac{z-i}{z-3}\right)=\frac{\pi}{6}$

**3.** **(d) :** $\sin^{-1}x = x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5}+\ldots$

$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots.$

$\therefore \quad \frac{2}{x^3}(\sin^{-1}x-\tan^{-1}x)=1-\frac{x^2}{4}+\ldots$

$\therefore \lim_{x\to 0} e^{-\frac{x^2}{4}\cdot\frac{2}{x^2}} = e^{\lambda}, \Rightarrow \lambda=-\frac{1}{2}$

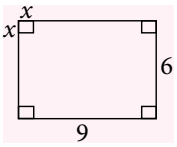
**4.** **(b) :** $S=\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots+\binom{11}{3}$

= coefficient of $x^3$ in $(1+x)^3+(1+x)^4+\ldots$
$\ldots+(1+x)^{11}$

= coefficient of $x^4$ in $(1+x)^3[(1+x)^9-1] = \binom{12}{4}$

= 495, with digit sum 18

**5.** **(a) :** $V(x) = x(9-2x)(6-2x)$




$\frac{dV}{dx}=0 \Rightarrow x=\frac{5-\sqrt{7}}{2}$

Max. Volume = $10+7\sqrt{7}$

**6.** **(a, d) :** $\begin{vmatrix}-1 & 1 & 1\\ 1 & 1 & k\\ -k & 2 & 1\end{vmatrix}=0 \Rightarrow k=0,3$

$k=0 \Rightarrow$ plane $\begin{vmatrix}x-2 & y-3 & z-4\\ 1 & 1 & 0\\ 0 & 2 & 1\end{vmatrix}=0$

$\Rightarrow x-y+2x=7$

$k=3 \Rightarrow$ plane $\begin{vmatrix}x-2 & y-3 & z-4\\ 1 & 1 & 3\\ -3 & 2 & 1\end{vmatrix}=0$

$\Rightarrow x+2y-z=4$

$\therefore$ Distances of the origin from planes

$= \frac{7}{\sqrt{6}}, \frac{4}{\sqrt{6}}$

**7.** **(d) :** Tangent to ellipse is $y=mx+\sqrt{4m^2+1}$

It is also tangent to circle

$\Rightarrow \sqrt{4m^2+1}=\sqrt{2}\sqrt{m^2+1} \Rightarrow |m| = \frac{1}{\sqrt{2}}$

**8.** **(a) :** $m=-\frac{1}{\sqrt{2}}$, the common tangent is

$x+\sqrt{2}y=\sqrt{6}$ touching ellipse $P\left(2\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\therefore$ Length of common tangent = $\sqrt{OP^2-2}=1$

**9.** **(7) :** $N$ = coefficient of $x^{12}$ in $(x+x^2+x^3+x^4)^8$

= coefficient of $x^4$ in $(1-x^4)^8(1-x)^{-8}$

$= \binom{11}{4}-8$

= 322 with digit sum 7.

**10.** **(b) :** $C_1$ has radius $R$ and Centre $S$

$C_2$ has radius $\frac{R}{2}$ and centre $N$, the mid point of orthocentre $O$ and circumcentre $S$

$OS = R\sqrt{3+2\Sigma\cos 2A}$

(P) $R-\frac{R}{2}=\frac{R}{2}\sqrt{3+2\Sigma\cos 2A} \Rightarrow \Sigma\cos 2A=-1$

(Q) $R^2+\frac{R^2}{4}=\frac{R^2}{4}(3+2\Sigma\cos 2A) \Rightarrow \Sigma\cos 2A = 1$

(R) $NS = R = \frac{R}{2}\sqrt{3+2\Sigma\cos 2A} \Rightarrow \Sigma\cos 2A=\frac{1}{2}$

(S) $R^2=\frac{R^2}{4}+\frac{R^2}{4}(3+2\Sigma\cos 2A) \Rightarrow \Sigma\cos 2A = 0$

### Solution Sender of Maths Musing

#### SET-153

1. Satya Dev - Bangalore
2. N. Jayanthi - Hyderabad
3. Gajula.Ravinder - Karimnagar
4. Gouri Sankar Adhokary, Subha Ganguly - West Bengal.

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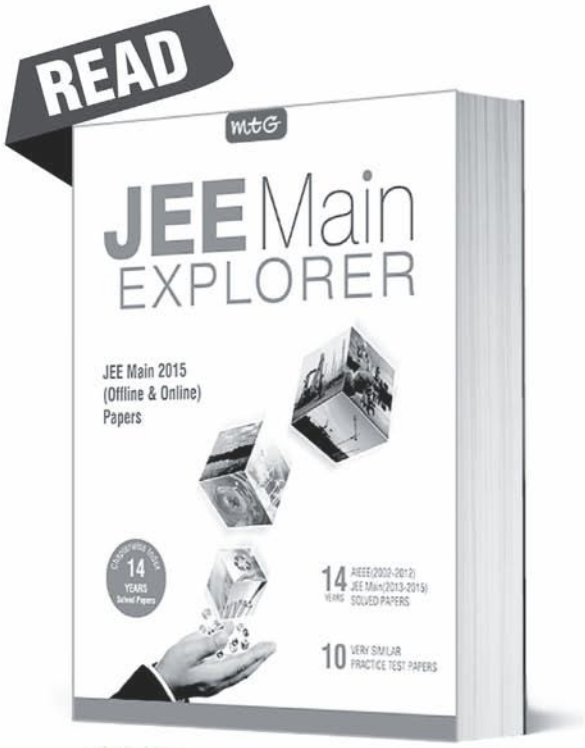


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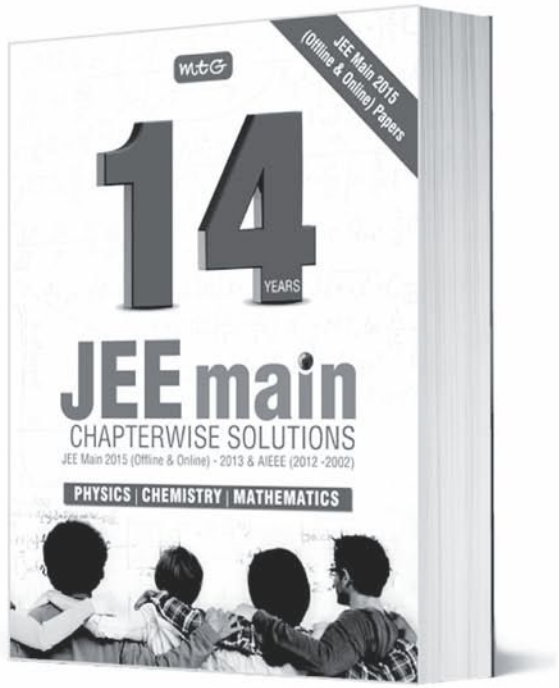


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# OLYMPIAD CORNER

1. Let $a$, $b$ and $c$ be positive real numbers such that $abc = 1$. Prove that
$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\le 1.$$
When does equality hold?

2. Let $f$ be a function from the set of real numbers $R$ into itself such that for all $x \in R$, we have $|f(x)| \le 1$ and $f\left(x+\frac{13}{42}\right)+f(x)=f\left(x+\frac{1}{6}\right)+f\left(x+\frac{1}{7}\right)$.
Prove that $f$ is a periodic function (that is, there exists a non-zero real number $c$ such that $f(x + c) = f(x)$ for all $x \in R$).

3. You have an unlimited supply of 5-gram and 8-gram weights that may be used in a pan balance. If you use only these weights and place them only in one pan, then find the largest number of grams that you cannot weigh.

4. The surface area of a closed cylinder is twice the volume. Determine the radius and height of the cylinder given that the radius and height are both integers.

5. It is given that cos $x$ = cos $y$ and sin $x$ = – sin $y$. Prove that sin 1994$x$ + sin 1994$y$ = 0.

## SOLUTIONS

1. We note first that
$$\frac{a^5+b^5}{2}\ge\left(\frac{a^3+b^3}{2}\right)\left(\frac{a^2+b^2}{2}\right),$$
Since $a^5 - a^3b^2 - a^2b^3 + b^5$
$= (a - b)^2 (a + b) (a^2 + ab + b^2) \ge 0$ with equality if and only if $a = b$.
Similarly
$$\frac{a^3+b^3}{2}\ge\left(\frac{a+b}{2}\right)\left(\frac{a^2+b^2}{2}\right)$$
because $a^3 - a^2b - ab^2 + b^3 = (a - b)^2 (a + b) \ge 0$, with equality if and only if $a = b$.

$$\text{Thus } \frac{a^5+b^5}{2} \ge \left(\frac{a^3+b^3}{2}\right)\left(\frac{a^2+b^2}{2}\right)\ge ab\left(\frac{a^3+b^3}{2}\right)$$
$$\ge ab\left(\frac{a+b}{2}\right)\left(\frac{a^2+b^2}{2}\right)\ge\frac{a^2b^2(a+b)}{2}.$$

It is enough, therefore, to prove
$$\frac{ab}{ab(a+b)ab+ab}+\frac{bc}{bc(b+c)bc+bc}+\frac{ca}{ca(c+a)ca+ca}\le 1,$$
$$\text{or } \frac{1}{ab(a+b)+abc}+\frac{1}{bc(b+c)+abc}+\frac{1}{ca(c+a)+abc}\le 1.$$
Equivalently,
$$\frac{1}{ab(a+b+c)}+\frac{1}{bc(a+b+c)}+\frac{1}{ca(a+b+c)}\le 1,$$
$$\text{or } \frac{c}{abc(a+b+c)}+\frac{a}{abc(a+b+c)}+\frac{b}{abc(a+b+c)}\le 1.$$
Again, because $abc = 1$, we get
$$\frac{a+b+c}{a+b+c} \le 1, \text{ which is true.}$$
The equality requires $a = b = c = 1$.

2. We will prove that $f$ is 1-periodic. We have
$$f\left(x+\frac{k}{6}+\frac{l+1}{7}\right)+f\left(x+\frac{k+1}{6}+\frac{l}{7}\right)$$
$$=f\left(x+\frac{k}{6}+\frac{l}{7}\right)+f\left(x+\frac{k+1}{6}+\frac{l+1}{7}\right).$$
If $k$ runs through 1, 2, . . . , $m$ – 1, where $m \in N$, and adding these equations, we obtain
$$f\left(x+\frac{l+1}{7}\right)+f\left(x+\frac{m}{6}+\frac{l}{7}\right)=f\left(x+\frac{m}{6}\right)+f\left(x+\frac{m}{6}+\frac{l+1}{7}\right).$$

Similarly when $l$ runs 1, 2, . . . , $n-1$, ($n \in N$) and adding these equations, we obtain

$$f\left(x+\frac{n}{7}\right)+f\left(x+\frac{m}{6}\right)=f(x)+f\left(x+\frac{m}{6}+\frac{n}{7}\right).$$

We choose $n = 7$ and $m = 6$, and find

$2f(x + 1) = f(x) + f(x + 2)$.

This means that the sequence $f(x + n)$ is an arithmetic sequence with common difference $f(x + 1) - f(x)$. But, since $f$ is bounded, we must have $f(x + 1) - f(x) = 0$. Hence 1 is a period of $f$. Finally, 1 is the "best" period because the function

$$f(x)=\frac{\{6x\}+\{7x\}}{2}$$

satisfies all the hypothesis of the problem.

**3.** If $W$ is the weight in grams then $W$ can be weighed if $W = 5x + 8y$, where $x$ and $y$ are non-negative integers. Furthermore, $y$ can be written in the form $5z + r$, where $z$ is a non -negative integer and $r$ is either 0, 1, 2, 3 or 4. Thus $W = 5(x + 8z) + 8(0)$, $W = 5(x + 8z) + 8(1)$, $W = 5(x + 8z) + 8(2)$, $W = 5(x + 8z) + 8(3)$, or $W = 5(x + 8z) + 8(5)$. These can be simplified to $W = 5k$, $W = 5(k + 1) + 3$, $W = 5(k + 3) + 1$, $W = 5(k + 4) + 4$ and $W = 5(k + 6) + 2$. By putting $k = x + 8z$, we conclude that $W$ can be weighed if $W = 5k + 8r$, with $k \geq 0$ and $r \leq 4$. Thus, according to the value of $r$, $W = 5k$, $W = 5(k + 1) + 3$, $W = 5(k + 3) + 1$, $W = 5(k + 4) + 4$, $W = 5(k + 6) + 2$. On the other hand, $W$ (as well as every integer) can take exactly one of the forms: $5K$, $5K + 1$, $5K + 2$, $5K + 3$, or $5K + 4$. By comparing these forms with the expressions for $W$ that can be weighed, we conclude:

(i) if $W$ is of the form $5K$, then it can be weighed;

(ii) if $W$ is of the form $5K + 1$, then it can be weighed if $K = k + 3$; that is, when $K \geq 3$;

(iii) if $W$ is of the form $5K + 2$, then it can be weighed if $K = k + 6$; that is, when $K \geq 6$;

(iv) if $W$ is of the form $5K + 3$, then it can be weighed if $K = k + 1$; that is, when $K \geq 1$;

(v) if $W$ is of the form $5K + 4$, then it can be weighed if $K = k + 4$; that is, when $K \geq 4$.

Consequently, the largest values of $W$ that cannot be weighed in categories: (ii), (iii), (iv) and (v), are $5(2) + 1 = 11$, $5(5) + 2 = 27$, $5(0) + 3 = 3$, $5(3) + 4 = 19$, respectively. The largest of them is 27.

**4.** Let $r$ be the radius and $h$ be the height of the closed cylinder, where $r$ and $h$ both are positive integers. If the surface area of the closed cylinder is twice the volume, then

$$2\pi rh + 2\pi r^2 = 2\pi r^2 h.$$

It follows that $h + r = rh$. Thus

$(r - 1)(h - 1) = rh - r - h + 1 = 0 + 1 = 1$.

Hence $r - 1 = 1$ and $h - 1 = 1$ (since $r$ and $h$ both are positive integers), so that $r = 2$ and $h = 2$.

It is easily checked that if $r = 2$ and $h = 2$, then the surface area of the closed cylinder is twice the volume.

**5.** More generally, we prove that

$\sin mx + \sin my = 0$, where $m$ is an integer.

Since $\sin mx + \sin my = 2\sin\left(m\frac{x+y}{2}\right) \times \cos\left(m\left(\frac{x-y}{2}\right)\right)$, it is sufficient to show that

$\sin\left(\frac{x+y}{2}\right)=0$. Since $\sin\left(m\frac{x+y}{2}\right)=0$ follows easily by induction from

$$\sin\left((m+1)\frac{x+y}{2}\right)=\sin\left(m\frac{(x+y)}{2}\right)\cos\left(\frac{x+y}{2}\right)+\cos\left(m\frac{(x+y)}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

Now,

$$\cos x = \cos y \Leftrightarrow \cos x - \cos y = 0$$

$$\Leftrightarrow -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}=0 \quad ...(1)$$

and $\sin x = -\sin y \Leftrightarrow \sin x + \sin y = 0$

$$\Leftrightarrow 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=0 \quad ...(2)$$

Squaring (1) and (2) and then adding, we find

$$4\sin^2\left(\frac{x+y}{2}\right)\underbrace{\left(\sin^2\frac{(x-y)}{2}+\cos^2\frac{(x-y)}{2}\right)}_{=1}=0$$

Hence $\sin\frac{x+y}{2}=0$.

# YOU ASK WE ANSWER

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**1. If $\frac{\sin^4\alpha}{a}+\frac{\cos^4\alpha}{b}=\frac{1}{a+b}$, then show that $\frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{1}{(a+b)^3}$.**

*–Reema Goyal, Lucknow*

**Ans.** Here $\frac{a+b}{a}\sin^4\alpha+\frac{a+b}{b}\cos^4\alpha=1$

or $\sin^4\alpha+\cos^4\alpha+\frac{b}{a}\sin^4\alpha+\frac{a}{b}\cos^4\alpha=1$

or $(\sin^2\alpha+\cos^2\alpha)^2-2\sin^2\alpha\cdot\cos^2\alpha+\frac{b}{a}\sin^4\alpha+\frac{a}{b}\cos^4\alpha=1$

or $\left(\sqrt{\frac{b}{a}}\sin^2\alpha-\sqrt{\frac{a}{b}}\cos^2\alpha\right)^2=0$

$\therefore\quad \sqrt{\frac{b}{a}}\sin^2\alpha=\sqrt{\frac{a}{b}}\cos^2\alpha$

$\Rightarrow\quad \frac{\sin^2\alpha}{a}=\frac{\cos^2\alpha}{b}=\frac{\sin^2\alpha+\cos^2\alpha}{a+b}$

$\Rightarrow\quad \sin^2\alpha=\frac{a}{a+b},\cos^2\alpha=\frac{b}{a+b}$

$\therefore\quad \frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{1}{a^3}\cdot\frac{a^4}{(a+b)^4}+\frac{1}{b^3}\cdot\frac{b^4}{(a+b)^4}$

$=\frac{a+b}{(a+b)^4}=\frac{1}{(a+b)^3}$.

**2. The ordinates of points $P$ and $Q$ on the parabola $y^2 = 12x$ are in the ratio 1:2. Find the locus of the point of intersection of the normals to the parabola at $P$ and $Q$.**

*–Mukesh Khanna, Muzaffarnagar*

**Ans.** Let coordinates of points $P$ and $Q$ respectively are $(3t_1^2, 6t_1)$ and $(3t_2^2, 6t_2)$. According to the given condition, we have

$\frac{6t_1}{6t_2}=\frac{1}{2}\Rightarrow t_2=2t_1$

Equation of the normals at $P$ and $Q$, respectively are $y=-t_1x+2at_1+at_1^3$ and $y=-t_2x+2at_2+at_2^3$.

If $R(h,k)$ be the intersection point of these normals, then we have

$k=-t_1h++2at_1+at_1^3$ ...(1)

and $k=-t_2h+2at_2+at_2^3$ ...(2)

Putting $t_2 = 2t_1$ in equation (2), we have

$\frac{k}{2}=-t_1h+2at_1+4at_1^3$ ...(3)

Subtracting equation (3) from equation (1), we have

$\frac{k}{2}=-3at_1^3$ *i.e.* $t_1=-\left(\frac{k}{6a}\right)^{1/3}$

Putting the value of $t_1$ in equation (1), we have

$k=(h-2a)\left(\frac{k}{6a}\right)^{1/3}-\frac{k}{6}\Rightarrow\left(\frac{7k}{6}\right)^3=(h-2a)^3\left(\frac{k}{6a}\right)$

*i.e.* $343k^2 = 12\,(h-6)^3$.

Hence, locus of $(h, k)$ is

$343y^2 = 12(x-6)^3$.

**3. A variable plane makes with the coordinate planes, a tetrahdron of constant volume $64k^3$. Then find the locus of the centroid of tetrahedron.**

*–Niyati Shukla (U.P.)*

**Ans.** Let the equation of the variable plane be

$lx + my + nz = p$

$\therefore$ Coordinates of the vertices of tetrahedron are

$(0,0,0);\left(\frac{p}{l},0,0\right);\left(0,\frac{p}{m},0\right)$ and $\left(0,0,\frac{p}{n}\right)$.

If $(x_1, y_1, z_1)$ be the centroid of tetrahedron, then

$x_1=\frac{p}{4l},y_1=\frac{p}{4m},z_1=\frac{p}{4n}$

$\therefore\quad \frac{p}{l}=4x_1,\frac{p}{m}=4y_1,\frac{p}{n}=4z_1$

Also, volume of tetrahedron $= 64k^3$

*i.e.*, $\frac{1}{6}\begin{vmatrix} p/l & 0 & 0\\ 0 & p/m & 0\\ 0 & 0 & p/n\end{vmatrix}=64k^3$

$\Rightarrow\quad \frac{1}{6}\begin{vmatrix} 4x_1 & 0 & 0\\ 0 & 4y_1 & 0\\ 0 & 0 & 4z_1\end{vmatrix}=64k^3$

$\Rightarrow\quad \frac{64x_1y_1z_1}{6}=64k^3\Rightarrow x_1y_1z_1=6k^3$

Hence, Locus of centroid is $xyz = 6k^3$.

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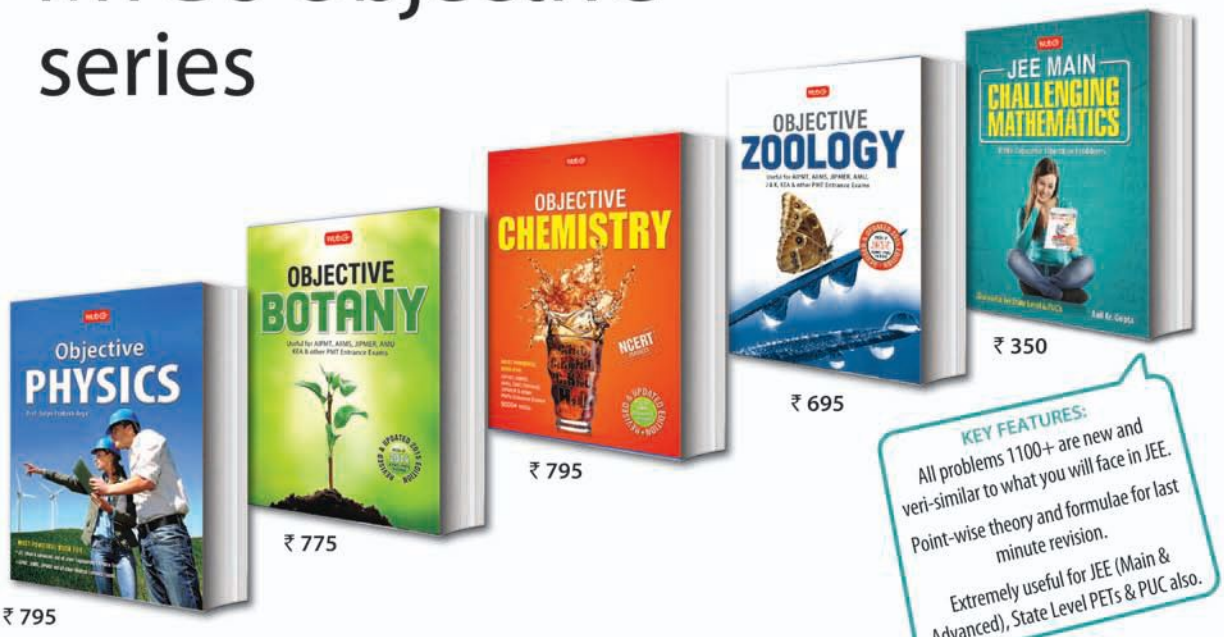


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**1.** A train travels at a speed of 10 m/s leaves Delhi for Punjab at 8 AM. At what distance from Punjab will it meet a train which leaves Punjab for Delhi at 8:20 AM and travels one-third faster than the former does, the distance from Delhi to Punjab being 68 km?
(a) 26 km (b) 32 km (c) 46 km (d) 48 km

**2.** A rectangular water tank is 80 m × 40 m. Water flows into it through a pipe 40 $cm^2$ at the opening at a speed of 10 km/hr. By how much, the water level will rise in the tank in half an hour?
(a) $\frac{3}{2}$cm (b) $\frac{4}{9}$cm
(c) $\frac{5}{8}$cm (d) None of these

**3.** Mr. Dayal invested money in two schemes A and B offering compound interest at 8% per annum and 9% per annum respectively. If the total amount of interest accrued through two schemes together in two years was ₹ 4818.30 and the total amount invested was ₹ 27000. What was the amount invested in scheme A.
(a) ₹ 12000 (b) ₹ 13500
(c) ₹ 15000 (d) None of these

**4.** If the number 653 $xy$ is divisible by 90, then $(x + y) = ?$
(a) 2 (b) 3 (c) 4 (d) 6

**5.** In a city, 40% of the people are illiterate and 60% are poor. Among the rich, 10% are illiterate. What percentage of the poor population is illiterate?
(a) 36% (b) 40%
(c) 60% (d) None of these

**6.** Two workers $A$ and $B$ are engaged to do a work. $A$ working alone takes 8 hours more to complete the job than if both worked together. If $B$ worked alone, he would need $4\frac{1}{2}$ hours more to complete the job than they both working together. What time would they take to do the work together?
(a) 4 hours (b) 5 hours
(c) 6 hours (d) 7 hours

**7.** Four different electronic devices make a beep after every 30 minutes, 1 hour, $1\frac{1}{2}$ hour and 1 hour 45 minutes respectively. All the devices beeped together at 12 noon. They will again beep together at :
(a) 12 midnight (b) 3 a.m.
(c) 6 a.m. (d) 9 a.m.

**8.** Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?
(a) 210 (b) 1050
(c) 25200 (d) 21400

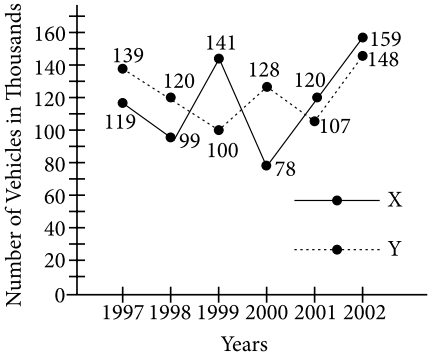
**9.** A man on the top of a vertical observation tower observes a car moving at a uniform speed coming directly towards it. If it takes 12 minutes for the angle of depression to change from 30° to 45°, how soon after this will the car reach the observation tower?
(a) 14 min. 35 sec (b) 15 min. 49 sec
(c) 16 min. 23 sec (d) 18 min. 5 sec

**10.** With a given rate of simple interest, the ratio of principal and amount for a certain period of time is 4 : 5. After 3 years, with the same rate of interest, the ratio of the principal and amount becomes 5 : 7. The rate of interest per annum is
(a) 4% (b) 5% (c) 6% (d) 7%

**11.** If $\tan\theta = \frac{1}{\sqrt{7}}$, then $\frac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta} = ?$
(a) $\frac{-3}{4}$ (b) $\frac{-2}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

**Direction (12-14) :**Study the following line-graph and answer the questions based on it.

Number of Vehicles Manufactured by Two Companies over the Years




**12.** What is the difference between the total productions of the two companies in the given years?
(a) 19000 (b) 22000 (c) 26000 (d) 28000

**13.** In which of the following years, the difference between the productions of companies X and Y was the maximum among the given years?
(a) 1997 (b) 1998 (c) 1999 (d) 2000

**14.** The production of Company Y in 2000 was approximately what percent of the production of Company X in the same year?
(a) 173 (b) 164 (c) 132 (d) 97

**15.** What is the unit digit in $(7^{95} - 3^{58})$?
(a) 0 (b) 4 (c) 6 (d) 7

**16.** If $a = b^x$, $b = c^y$ and $c = a^z$, then the value of $xyz$ is equal to
(a) –1 (b) 0 (c) 1 (d) $abc$

**17.** In a class, the number of girls is 20% more than that of the boys. The strength of the class is 66. If 4 more girls are admitted to the class, what will be the ratio of the number of boys to that of the girls?
(a) 1 : 2 (b) 1 : 4 (c) 3 : 4 (d) 3 : 5

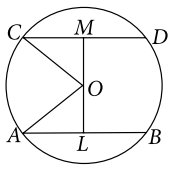
**18.** A man piles logs of wood so that the top layer contains one log and each lower layer has one more than the layer above. If there are 15 layers, the total number of logs will be
(a) 105 (b) 120 (c) 210 (d) 225

**19.** A box contains 10 black and 10 white balls. The probability of drawing two balls of the same colour, is :
(a) $\frac{9}{19}$ (b) $\frac{9}{38}$ (c) $\frac{10}{19}$ (d) $\frac{5}{19}$

**20.** What is the smallest number by which 3600 be divided to make it a perfect cube?
(a) 9 (b) 50 (c) 300 (d) 450

**21.** Rohit was asked to find $\frac{7}{9}$ of a fraction. But, he made a mistake of dividing the given fraction by $\frac{7}{9}$ and got an answer which exceeded the correct answer by $\frac{8}{21}$. The correct answer is
(a) $\frac{3}{7}$ (b) $\frac{7}{12}$ (c) $\frac{2}{21}$ (d) $\frac{1}{3}$

**22.** In the given figure (not drawn to scale) $AB$ and $CD$ are two parallel chords on the opposite sides of the centre of the circle. If $AB$ = 10 cm, $CD$ = 24 cm and the radius of the circle is 13 cm, the distance between the chords is




(a) 17 cm
(b) 15 cm
(c) 16 cm
(d) 18 cm

**23.** A man wants to sell his old scooter. He has two customers. One of them wants to pay ₹ 12000 in cash and the other in ₹ 13500 payable after 8 months. If the rate of interest is 12% per annum, which option is better for the seller?
(a) ₹ 12000 in cash
(b) ₹ 13500 payable after 8 months
(c) both are equally good
(d) none of these

**24.** $A$ and $B$ solved a quadratic equation. In solving it, $A$ made a mistake in the constant term and obtained the roots as 6 and 2, while $B$ made a mistake in the coefficient ofr x only and obtained the roots as –7 and –1. Find the correct roots of the equation.
(a) 1, 7 (b) 6, –1 (c) 2, –7 (d) 8, 6

**25.** A person was asked to state his age in years. His reply was "Take my age three years hence, multiply it by 3 and then subtract three times my age three years ago and you will know how old I am". What was the age of the person?
(a) 18 years (b) 20 years
(c) 24 years (d) 32 years

## ANSWER KEYS

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** | (b) | **2.** | (c) | **3.** | (a) | **4.** | (c) | **5.** | (c) |
| **6.** | (c) | **7.** | (d) | **8.** | (c) | **9.** | (c) | **10.** | (b) |
| **11.** | (d) | **12.** | (c) | **13.** | (d) | **14.** | (b) | **15.** | (b) |
| **16.** | (c) | **17.** | (c) | **18.** | (b) | **19.** | (a) | **20.** | (d) |
| **21.** | (b) | **22.** | (a) | **23.** | (b) | **24.** | (a) | **25.** | (a) |

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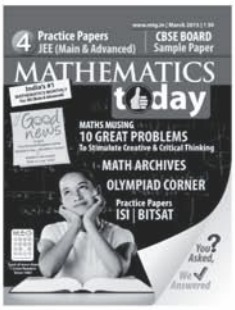

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